DEFORMATIONS OF TRIANGULINE B-PAIRS AND ZARISKI DENSITY OF TWO DIMENSIONAL CRYSTALLINE REPRESENTATIONS.

KENTARO NAKAMURA

Abstract. The aims of this article are to study deformation theory of trianguline B-pairs and to construct a p-adic family of two dimensional trianguline representations for any p-adic field. The deformation theory is the generalization of Bellaïche-Chenevier's and Chenevier's works in the $\mathbb{Q}_p$-case, where they used (φ, Γ)-modules over the Robba ring instead of using B-pairs. Generalizing and modifying Kisin's theory of X_{fs} for any p-adic field, we construct a p-adic family of two dimensional trianguline representations. As an application of these theories, we prove a theorem concerning Zariski density of two dimensional crystalline representations for any p-adic field, which is a generalization of Colmez's and Kisin's theorem for $\mathbb{Q}_p$-case.

Contents

1. Introduction.

1.1. **Background.** Let p be a prime number and K be a p-adic field, i.e. finite extension of $\mathbb{Q}_p$. The theory of trianguline representations (which form a class of p-adic representations of $G_K := \mathrm{Gal}(\overline{K}/K)$), in particular the theory of their p-adic families turns out to be very important in the study of p-adic Galois representations parametrized by p-adic families of automorphic forms (in particular eigenvarieties). Inspired by Kisin's p-adic Hodge theoretic study of Coleman-Mazur

2008 Mathematical Subject Classification 11F80 (primary), 11F85, 11S25 (secondary). Keywords: p-adic Hodge theory, trianguline representations, B-pairs.

eigencurve ([Ki03]), Colmez ([Co08]) defined the notion of trianguline representations by using Fontaine's and Kedlaya's theory of (φ, Γ)-modules over the Robba ring in the study of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Based on their works, Bellaïche-Chenevier ([Bel-Ch09]) and Chenevier ([Ch09b]) studied deformation theory of trianguline representations and p-adic families of trianguline representations. These theories are the most fundamental tools for their study of unitary eigenvarieties. Because all their studies are limited to the case $K = \mathbb{Q}_p$, we didn't have any results concerning to p-adic Hodge theoretic properties of eigenvarieties over a number field F except when F is $\mathbb{Q}$ or more generally is a number field in which p splits completely.

On the other hands, in [Na09], the author of this article generalized many results of [Co08] for any p-adic field K. The author proved some fundamental properties of trianguline representations and then classified two dimensional trianguline representations for any p-adic field, where we studied trianguline representations by using B-pairs, which was defined by Berger in [Be09], instead of using (φ, Γ)-modules over the Robba ring.

The aim of this article is to generalize Kisin's, Bellaïche-Chenevier's and Chenevier's works for any p-adic field K, more precisely, to develop deformation theory of trianguline representations and to construct a p-adic family of two dimensional trianguline representations for any p-adic field K. The author thinks that these generalizations are fundamental for applications to p-adic Hodge theoretic study of eigenvarieties for more general number fields. Moreover, as an application of these theories, we prove some theorems (see Theorem 1.6 and Theorem1.7 in Introduction) concerning Zariski density of two dimensional crystalline representations for any p-adic field. These results are generalizations of a theorem of Colmez and Kisin when $K = \mathbb{Q}_p$, which played some crucial roles in the proof of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Co10], [Ki10], [Pa10]).

In the next article ([Na11]) which is based on many results of this article, we construct a p-adic family of d-dimensional trianguline representations for any $d \in \mathbb{Z}_{\geqq 1}$ and for any K and prove a theorem concerning to Zariski density of d-dimensional crystalline representations for any d and K.

In future works, the author wants to study eigenvarieties which parametrize finite slope p-adic Hilbert modular forms by using the results of this article.

1.2. **Overview.** Here, we explain the details of this article.

In § 2, we study the deformation theory of trianguline B-pairs, which is the generalization of the studies of [Bel-Ch09], [Ch09b] for any p-adic field.

In § 2.1, we recall the definition of B-pairs and some fundamental properties of trianguline B-pairs proved in [Na09] and then we extend these to Artin local ring coefficients case. Let E be a suitable finite extension of $\mathbb{Q}_p$ as in Notation below. We recall the definition of E-B-pairs of G_K, which is the E-coefficient version of B-pairs. We write $B_e := B_{\mathrm{cris}}^{\varphi=1}$. An E-B-pair is a pair $W = (W_e, W^+_{\mathrm{dR}})$ where

W_e is a finite free $B_e \otimes_{\mathbb{Q}_p} E$-module with a continuous semi-linear G_K-action such that $W^+_{\mathrm{dR}} \subseteq W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$ is a G_K-stable $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-lattice of W_{dR}. The category of E-representations of G_K is embedded in the category of E-B-pairs by $V \mapsto W(V) := (B_e \otimes_{\mathbb{Q}_p} V, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. We say that an E-B-pair is split trianguline if W is a successive extension of rank one E-B-pairs, i.e. W has a filtration $0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ such that W_i is a saturated sub E-B-pair of W and W_i/W_{i-1} is a rank one E-B-pair for any $1 \leqq i \leqq n$. We say that W is trianguline if $W \otimes_E E'$ is a E'-split trianguline E'-B-pair for a finite extension E' of E. We say that an E-representation V is split trianguline (resp. trianguline) if $W(V)$ is split trianguline (resp. trianguline). By these definitions, to study trianguline E-B-pairs, we first need to classify rank one E-B-pairs and then we need to calculate extension class group of them, which were studied in [Co08] for $K = \mathbb{Q}_p$ and in [Na09] for general K. In § 2.1, we recall these results which we need to study the deformation theory of trianguline E-B-pairs. We define the Artin local ring coefficient version of B-pairs. Let $\mathcal{C}_E$ be the category of Artin local E-algebras with the residue field isomorphic to E. For any $A \in \mathcal{C}_E$, we say that $W := (W_e, W^+_{\mathrm{dR}})$ is an A-B-pair if W_e is a finite free $B_e \otimes_{\mathbb{Q}_p} A$-module with a continuous semi-linear G_K-action and $W^+_{\mathrm{dR}} \subseteq W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$ is a G_K-stable $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-lattice. We generalize some results of [Na09] for A-B-pairs.

In § 2, we study two types of deformations of split trianguline E-B-pairs. In § 2.2, first we study the usual deformation for any E-B-pairs which is the generalization of Mazur's deformation theory of p-adic Galois representations. Let W be an E-B-pair and $A \in \mathcal{C}_E$. We say that (W_A, ι) is a deformation of W over A if W_A is an A-B-pair and $\iota : W_A \otimes_A E \xrightarrow{\sim} W$ is an isomorphism. We define the deformation functor of W, $D_W : \mathcal{C}_E \to (\mathrm{Sets})$ by $D_W(A) := \{$equivalent classes of deformations (W_A, ι) of W over A $\}$. We prove the following proposition concerning to pro-representability and formally smoothness and dimension formula of D_W.

Proposition 1.1. *(Corollary 2.30) Let W be an E-B-pair of rank n. If W satisfies the following conditions,*

(1) $\mathrm{End}_{G_K}(W) = E$,
(2) $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$,

then the functor D_W is pro-representable by R_W such that

$$R_W \xrightarrow{\sim} E[[T_1, \cdots, T_d]] \quad \text{where } d := [K : \mathbb{Q}_p]n^2 + 1.$$

In § 2.3, we study the other more important type of deformation, i.e. trianguline deformation. Let W be a split trianguline E-B-pair of rank n and $\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ be a fixed triangulation of W. For any $A \in \mathcal{C}_E$, we say that $(W_A, \iota, \mathcal{T}_A)$ is a trianguline deformation of $(W, \mathcal{T})$ over A if (W_A, ι) is a deformation of W over A and $\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n-1,A} \subseteq W_{n,A} = W_A$ is an A-triangulation of W_A (i.e. $W_{i,A}$ is a saturated sub A-B-pair of W_A such

that $W_{i,A}/W_{i-1,A}$ is a rank one A-B-pair for any i) such that $\iota(W_{i,A}\otimes_A E)=W_i$ for any i. We define the trianguline deformation functor $D_{W,\mathcal{T}}:\mathcal{C}_E\rightarrow$ (Sets) of $(W,\mathcal{T})$ by $D_{W,\mathcal{T}}(A):=\{$equivalent classes of trianguline deformations $(W_A,\iota,\mathcal{T}_A)$ of $(W,\mathcal{T})$ over $A\}$. We prove the following proposition concerning to the pro-representability and formally smoothness and dimension formula of this functor, which is a generalization of Proposition 3.6 of [Ch09b] for any p-adic field.

Proposition 1.2. *(Proposition 2.40) Let W be a split trianguline E-B-pair of rank n with a triangulation $\mathcal{T}:0\subseteq W_1\subseteq\cdots\subseteq W_{n-1}\subseteq W_n=W$. We assume that $(W,\mathcal{T})$ satisfies the following conditions,*

(0) $\mathrm{End}_{G_K}(W)=E$,
(1) *For any $1\leqq i<j\leqq n$, $\delta_j/\delta_i\neq\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma\in\mathcal{P}}\in\prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\leqq 0}$*
(2) *For any $1\leqq i<j\leqq n$, $\delta_i/\delta_j\neq|\mathrm{N}_{K/\mathbb{Q}_p}|\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma\in\mathcal{P}}\in\prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\geqq 1}$,*

then $D_{W,\mathcal{T}}$ is pro-represented by a quotient ring $R_{W,\mathcal{T}}$ of R_W such that

$$R_{W,\mathcal{T}}\xrightarrow{\sim}E[[T_1,\cdots,T_{d_n}]]\ \textit{where}\ d_n:=\frac{n(n+1)}{2}[K:\mathbb{Q}_p]+1.$$

In § 2.4, we define the notion of benign E-B-pairs, which forms a special good class of split trianguline and potentially crystalline E-B-pairs, and prove a theorem concerning to tangent spaces of the deformation rings of this class. In § 2.4.1, we define the notion of benign E-B-pairs , in [Ch09b] this class is called generic, in this article we follow the terminology of [Ki10]. Let W be a potentially crystalline E-B-pair of rank n such that $W|_{G_L}$ is crystalline for a finite totally ramified abel extension L of K (we call such a representation a crystabelline representation). We assume that $D^L_{\mathrm{cris}}(W):=(B_{\mathrm{cris}}\otimes_{B_e}W_e)^{G_L}=K_0\otimes_{\mathbb{Q}_p}Ee_1\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_n$ such that $K_0\otimes_{\mathbb{Q}_p}Ee_i$ are preserved by $(\varphi,\mathrm{Gal}(L/K))$-action and that $\varphi^f(e_i)=\alpha_ie_i$ for some $\alpha_i\in E^{\times}$, here $f:=[K_0:\mathbb{Q}_p]$ and K_0 is the maximal unramified extension of $\mathbb{Q}_p$ in K. We denote by $\{k_{1,\sigma}.k_{2,\sigma},\cdots,k_{n,\sigma}\}_{\sigma:K\hookrightarrow\overline{K}}$ the Hodge-Tate weight of W such that $k_{1,\sigma}\geqq k_{2,\sigma}\geqq\cdots\geqq k_{n,\sigma}$ for any $\sigma:K\hookrightarrow\overline{K}$, in this paper we define the Hodge-Tate weight of $\mathbb{Q}_p(1)$ by 1. Let $\mathfrak{S}_n$ be the n-th permutation group. For any $\tau\in\mathfrak{S}_n$, we can define a filtration on $D^L_{\mathrm{cris}}(W)$ by $0\subseteq K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(1)}\subseteq K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(1)}\oplus K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(2)}\subseteq\cdots\subseteq K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(1)}\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(n-1)}\subseteq D^L_{\mathrm{cris}}(W)$ by sub E-filtered (φ,G_K)-modules, where the filtration of $K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(1)}\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(i)}$ is the one induced from that of $D^L_{\mathrm{dR}}(W)=L\otimes_{K_0}D^L_{\mathrm{cris}}(W)$. By the equivalence between the category of potentially crystalline B-pairs and the category of filtered (φ,G_K)-modules, for any $\tau\in\mathfrak{S}_n$ we obtain the triangulation $\mathcal{T}_\tau:0\subseteq W_{\tau,1}\subseteq W_{\tau,2}\subseteq\cdots\subseteq W_{\tau,n}=W$ such that $W_{\tau,i}$ is potentially crystalline and $D^L_{\mathrm{cris}}(W_{\tau,i})\xrightarrow{\sim}K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(1)}\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(i)}$ for any $1\leqq i\leqq n$.

Under this situation, we define the notion of benign E-B-pairs as follows.

Definition 1.3. Let W be a potentially crystalline E-B-pair of rank n as above. Then we say that W is benign if W satisfies the following;

(1) For any $i \neq j$, we have $\alpha_i/\alpha_j \neq 1, p^f, p^{-f}$,
(2) For any $\sigma : K \hookrightarrow \overline{K}$, we have $k_{1,\sigma} > k_{2,\sigma} > \cdots > k_{n-1,\sigma} > k_{n,\sigma}$,
(3) For any $\tau \in \mathfrak{S}_n$ and $1 \leqq i \leqq n$, the Hodge-Tate weight of $W_{\tau,i}$ is $\{k_{1,\sigma}, k_{2,\sigma}, \cdots, k_{i,\sigma}\}_{\sigma:K\hookrightarrow\overline{K}}$.

In § 2.4.2, we prove the main theorem of § 2. Let W be a benign E-B-pair of rank n as above. For any $\tau \in \mathfrak{S}_n$, we can define the trianguline deformation functor $D_{W,\mathcal{T}_\tau}$. Let R_W be the universal deformation ring of D_W and $R_{W,\mathcal{T}_\tau}$ be the universal deformation ring of $D_{W,\mathcal{T}_\tau}$ for any $\tau \in \mathfrak{S}_n$. Let $t(R_W)$ and $t(R_{W,\mathcal{T}_\tau})$ be the tangent spaces of R_W and $R_{W,\mathcal{T}_\tau}$. For any $\tau \in \mathfrak{S}_n$, $t(R_{W,\mathcal{T}_\tau})$ is a sub E-vector space of $t(R_W)$. The main theorem of § 2 is the following, which is the generalization of Theorem 3.19 of [Ch09b] for any p-adic field.

Theorem 1.4. (*Theorem 2.61*) *Let W be a benign E-B-pair of rank n, then we have an equality*

$$\sum_{\tau\in\mathfrak{S}_n} t(R_{W,\mathcal{T}_\tau}) = t(R_W).$$

This theorem is a crucial local result for applications to some Zariski density theorems of local or global p-adic Galois representations. In fact, using this theorem for $K = \mathbb{Q}_p$, Chenevier ([Ch09b]) proved a theorem concerning to Zariski density of unitary automorphic Galois representations in universal deformation spaces of three dimensional self dual p-adic representations of G_F for any CM field F in which p splits completely. Moreover, this theorem is also a crucial result for the proof of Zariski density of crystalline representations in universal deformation spaces of p-adic Galois representations of p-adic fields. In the rest of this paper § 3 and §4, we apply this theorem only for the two dimensional case. Using this theorem for $K = \mathbb{Q}_p$, Chenevier ([Ch10]) recently proved Zariski density of crystalline representations for higher dimensional and $K = \mathbb{Q}_p$ case. In the next paper ([Na11]), the author uses this theorem for proving Zariski density of crystalline representations for higher dimensional and any p-adic field case.

In § 3, we construct p-adic families of two dimensional trianguline representations for any p-adic field by generalizing Kisin's theory of finite slope sub space X_{fs} in [Ki03] for any p-adic field. As in $\mathbb{Q}_p$ case of [Ki03], [Ki10], this family is essential for the proof of Zariski density of two dimensional crystalline representations in §4 and for p-adic Hodge theoretic studies of Hilbert modular eigenvarieties, which will be a subject in author's subsequent works.

In § 3.1, we prove some propositions concerning Banach G_K-modules which we need for constructing p-adic families of trianguline representations. In particular, we show that these Banach G_K-modules can be obtained naturally from some almost $\mathbb{C}_p$-representations ([Fo03]). For us, one of the important properties of these Banach G_K-modules is orthonormalizability as Banach modules over some Banach algebras. All these properties follow from some general facts of almost $\mathbb{C}_p$-representations.

In § 3.2, for any separated rigid analytic space X over E and for any finite free $\mathcal{O}_X$-module M with a continuous G_K-action and for any invertible function Y on X, we construct a Zariski closed sub rigid analytic space X_{fs} of X, which is "roughly" defined as the subspace of X consisting of points $x \in X$ such that $D^+_{\mathrm{cris}}(M(x))^{\varphi^f = Y(x)} \neq 0$, where $M(x)$ is the fiber of M at x and $f := [K_0 : \mathbb{Q}_p]$. For the precise characterization of X_{fs}, see Theorem 3.9. This construction is the generalization of Kisin's X_{fs} in §5 of [Ki03] for any p-adic field. After obtaining the results in § 3.1, the construction and the proof is almost all the same as that of [Ki03], a difference is that we need to consider all the embeddings $\tau : K \hookrightarrow \overline{K}$. But, for convenience of readers or the author, we choose to re-prove this construction in full detail.

In § 3.3, we apply this construction to the rigid analytic spaces associated to the universal deformation ring of two dimensional mod p-representations of G_K. Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a two dimensional mod p representation of G_K, where $\mathbb{F}$ is the residue field of E. For simplicity, in this paper, we assume that $\mathrm{End}_{\mathbb{F}[G_K]}(\bar{\rho}) = \mathbb{F}$, then there exists the universal deformation ring $R_{\bar{\rho}}$ of $\bar{\rho}$, which is a local complete noetherian $\mathcal{O}$-algebra, where $\mathcal{O}$ is the integer ring of E. Let $\mathfrak{X}(\bar{\rho})$ be the rigid analytic space over E associated to $R_{\bar{\rho}}$. The universal deformation V^{univ} of $\bar{\rho}$ over $R_{\bar{\rho}}$ defines a rank two free $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-module $\tilde{V}^{\mathrm{univ}}$ with a continuous $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-linear G_K-action. $\mathfrak{X}(\bar{\rho})$ parametrizes p-adic representations V of G_K whose reductions are isomorphic to $\bar{\rho}$ for some G_K-stable lattices of V. Let $\mathcal{W}$ be the rigid analytic space over E which represents the functor $D_{\mathcal{W}} : \{$ rigid analytic spaces over $E\} \to$ (Sets) defined by $D_{\mathcal{W}}(X) := \{\delta : \mathcal{O}_K^\times \to \mathcal{O}_X^\times :$ continuous homomorphisms $\}$ for any rigid analytic space X over E. Let $\delta^{\mathrm{univ}} : \mathcal{O}_K^\times \to \mathcal{O}_{\mathcal{W}}^\times$ be the universal homomorphism. If we fix a uniformizer $\pi_K \in K$, there exists unique character $\tilde{\delta}^{\mathrm{univ}} : G_K^{\mathrm{ab}} \to \mathcal{O}_{\mathcal{W}}^\times$ such that $\tilde{\delta}^{\mathrm{univ}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^\times} = \delta^{\mathrm{univ}}$ and $\tilde{\delta}^{\mathrm{univ}} \circ \mathrm{rec}_K(\pi_K) = 1$, where $\mathrm{rec}_K : K \hookrightarrow G_K^{\mathrm{ab}}$ is the reciprocity map of local class field theory. We denote by $X(\bar{\rho}) := \mathfrak{X}(\bar{\rho}) \times_E \mathcal{W} \times_E \mathbb{G}^{\mathrm{an}}_{m,E}$ and denote by $p_1 : X(\bar{\rho}) \to \mathfrak{X}(\bar{\rho})$, $p_2 : X(\bar{\rho}) \to \mathcal{W}$ and $p_3 : X(\bar{\rho}) \to \mathbb{G}^{\mathrm{an}}_{m,E}$ the canonical projections. For any $x \in X(\bar{\rho})$, we denote by $E(x)$ the residue field at x which is a finite extension of E. For any two dimensional trianguline representation $V \in \mathfrak{X}(\bar{\rho})$ with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W(V \otimes_E E')$ for some E', we define a point $x_{(V,\mathcal{T})} := (V, \delta_1|^\times_{\mathcal{O}_K}, \delta_1(\pi_K)) \in X(\bar{\rho})$. We define $M := p_1^*(\tilde{V}^{\mathrm{univ}})((p_2^*\tilde{\delta}^{\mathrm{univ}})^{-1})$ a rank two $\mathcal{O}_{X(\bar{\rho})}$-module with a continuous $\mathcal{O}_{X(\bar{\rho})}$-linear G_K-action. Let $Y := p_3^*(T) \in \mathcal{O}^\times_{X(\bar{\rho})}$ be the pullback of the canonical coordinate of $\mathbb{G}^{\mathrm{an}}_{m,E}$. If we apply the construction of X_{fs} to the triple $(X(\bar{\rho}), M, Y)$, we obtain a Zariski closed sub rigid analytic space $X(\bar{\rho})_{fs}$ of $X(\bar{\rho})$, which we denote by $\mathcal{E}(\bar{\rho}) := X(\bar{\rho})_{fs}$. The main result of § 3 is the following theorem (see Theorem 3.16 and Theorem 3.21 for more precise statements), which is a generalization of Proposition 10.4 and 10.6 of [Ki03].

Theorem 1.5. *$\mathcal{E}(\bar{\rho})$ satisfies the following properties.*

(1) *For any point $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$, V_x is a trianguline representation.*

(2) *Conversely, if* $x=[V_x]\in \mathfrak{X}(\bar{\rho})$ *is a point such that* V_x *is a split trianguline* $E(x)$*-representation with a triangulation* $\mathcal{T}: 0\subseteq W(\delta_1)\subseteq W(V_x)$ *satisfying all the conditions in Proposition 1.2, then the point* $x_{(V_x,\mathcal{T})}\in X(\bar{\rho})$ *defined above is contained in* $\mathcal{E}(\bar{\rho})$.

(3) *For any point* $x_{(V_x,\mathcal{T})}$ *as in* (2), *there exists an isomorphism* $\hat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x_{(V_x,\mathcal{T})}}\xrightarrow{\sim} R_{V_x,\mathcal{T}}$, *in particular* $\mathcal{E}(\bar{\rho})$ *is smooth of dimension* $3[K:\mathbb{Q}_p]+1$ *at such points.*

In the next paper ([Na11]), the author will generalize all these results for higher dimensional case.

In the final section § 4, as an application of § 2 (in the two dimensional case) and of § 3, we prove the following theorems concerning Zariski density of two dimensional crystalline representations. We denote by $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}:=\{x\in\mathfrak{X}(\bar{\rho})|V_x$ (the p-adic representation corresponding to x) is crystalline with Hodge-Tate weight $\{k_{1,\sigma},k_{2,\sigma}\}_{\sigma:K\hookrightarrow\overline{K}}$ such that $k_{1,\sigma}\neq k_{2,\sigma}$ for any $\sigma\}$, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}:=\{x\in\mathfrak{X}(\bar{\rho})|V_x$ is crystalline and benign $\}$. We denote by $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ the Zariski closure of $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ in $\mathfrak{X}(\bar{\rho})$.

Theorem 1.6. (*Theorem 4.14*) *If* $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is non empty, then* $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ *is non empty and a union of irreducible components of* $\mathfrak{X}(\bar{\rho})$.

We note that we can prove all the above theorems even if $\mathrm{End}_{G_K}(\bar{\rho})\neq\mathbb{F}$ without any additional difficulties using the universal framed deformations instead of using usual deformations.

Moreover, under the following assumptions, we prove the following stronger theorem.

Theorem 1.7. (*Theorem 4.16*) *We assume the following conditions,*

(0) $\mathrm{End}_{G_K}(\bar{\rho})=\mathbb{F}$,

(1) $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is not empty,*

(2) $\bar{\rho}\not\sim\begin{pmatrix}1 & * \\ 0 & \omega\end{pmatrix}\otimes\chi$ *for any* $\chi: G_K\rightarrow\mathbb{F}^{\times}$, *where* ω *is the mod* p *cyclotomic character,*

(3) $[K(\zeta_p):K]\neq 2$ *or, for any* χ, $\bar{\rho}|_{I_K}\not\sim\begin{pmatrix}\chi_2^i & 0 \\ 0 & \chi_2^{ip^f}\end{pmatrix}\otimes\chi$ *such that* $i=\frac{p^f+1}{2}$, *where* ζ_p *is a primitive* p*-th root of* 1 *and* $\chi_2: I_K\rightarrow\mathbb{F}^{\times}$ *is a fundamental character of the second kind,*

then we have an equality $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}=\mathfrak{X}(\bar{\rho})$.

This theorem generalizes the results of Colmez ([Co08]) and Kisin ([Ki10]) for any p-adic field case. As in stated in the above paragraph, Chenevier recently proved similar results for higher dimensional and for $K=\mathbb{Q}_p$ and the author will prove these theorems in full generality (i.e. for higher dimensional and for any p-adic field) in the next paper.

Notation. Let p be a prime number. K be a finite extension of $\mathbb{Q}_p$, $\overline{K}$ a fixed algebraic closure of K, K_0 the maximal unramified extension of $\mathbb{Q}_p$ in K, K^{nor} the Galois closure of K in $\overline{K}$. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K equipped with pro-finite topology. $\mathcal{O}_K$ is the ring of integers of K, $\pi_K \in \mathcal{O}_K$ is a uniformizer of K, $k := \mathcal{O}_K/\pi_K\mathcal{O}_K$ is the residue field of K, $q = p^f := \sharp k$ is the order of k, $\chi_p : G_K \to \mathbb{Z}_p^{\times}$ is the p-adic cyclotomic character (i.e. $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for any p^n-th roots of unity and for any $g \in G_K$). Let $\mathbb{C}_p := \hat{\overline{K}}$ be the p-adic completion of $\overline{K}$, which is an algebraically closed p-adically completed field, and $\mathcal{O}_{\mathbb{C}_p}$ be its ring of integers. We denote by v_p the normalized valuation on $\mathbb{C}_p^{\times}$ such that $v_p(p) = 1$. We denote by $|-|_p : \mathbb{C}_p \to \mathbb{R}_{\geqq 0}$ the absolute value such that $|p|_p = \frac{1}{p}$. Let E be a finite extension of $\mathbb{Q}_p$ in $\overline{K}$ such that $K^{\mathrm{nor}} \subseteq E$. In this paper, we use the notation E as a coefficient field of representations. We denote by $\mathcal{P} := \{\sigma : K \hookrightarrow \overline{K}\} = \{\sigma : K \hookrightarrow E\}$ the set of $\mathbb{Q}_p$-algebra homomorphisms from K to $\overline{K}$ (or E). Let $\chi_{\mathrm{LT}} : G_K \to \mathcal{O}_K^{\times} \hookrightarrow \mathcal{O}_E^{\times}$ be the Lubin-Tate character associated with the fixed uniformizer π_K. Let $\mathrm{rec}_K : K^{\times} \to G_K^{\mathrm{ab}}$ be the reciprocity map of local class field theory such that $\mathrm{rec}_K(\pi_K)$ is a lifting of the inverse of q-th power Frobenius on k, then $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K : K^{\times} \to \mathcal{O}_K^{\times}$ satisfies $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K(\pi_K) = 1$ and $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^{\times}} = \mathrm{id}_{\mathcal{O}_K^{\times}}$.

Acknowledgement. The author would like to thank Kenichi Bannai for reading the first version of this article and for helpful comments. He also would like to thank Tadashi Ochiai for many valuable and interesting discussions and also thank Yoichi Mieda, Atsushi Shiho and Takahiro Tsushima for discussing about rigid geometry which we use in § 4 and also thank Gaëtan Chenevier for discussing about generalization to higher dimensional case and also thank Go Yamashita and Seidai Yasuda for pointing out some mistakes in the previous version of this article and for many valuable discussions. The author is supported by KAKENHI (21674001). This research was also supported in part by the JSPS Core-to-Core program 28005 (Represented by Makoto Matsumoto). Part of this paper was written at l'Institute Henri Poincaré during Galois Trimester. The author would like to thank his host Pierre Colmez and all of the people at Keio University related to the JSPS ITP program which enabled my stay.

2. Deformation theory of trianguline B-pairs.

2.1. **Review of B-pairs.**

2.1.1. *E-B-pairs.* We start by recalling the definition of E-B-pairs ([Be09], [Na09]) and then recall fundamental properties of them established in [Na09]. First, we recall some p-adic period rings ([Fo94]) which we need for defining B-pairs. Let $\widetilde{\mathbb{E}}^+ := \varprojlim_n \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\sim} \varprojlim_n \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, where the limits are taken with respect to p-th power maps. It is known that $\widetilde{\mathbb{E}}^+$ is a complete valuation ring of characteristic p

whose valuation is defined by $v(x) := v_p(x^{(0)})$ (here $x = (x^{(n)})_{n\geqq 0} \in \varprojlim_n \mathcal{O}_{\mathbb{C}_p}$). We fix a system of p^n-th roots of unity $\{\varepsilon^{(n)}\}_{n\geqq 0}$ such that $\varepsilon^{(0)} = 1$, $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$, $\varepsilon^{(1)} \neq 1$. Then $\varepsilon := (\varepsilon^{(n)})$ is an element of $\widetilde{\mathbb{E}}^+$ such that $v(\varepsilon - 1) = p/(p-1)$. G_K acts on this ring continuously in natural way. We put $\widetilde{\mathbb{A}}^+ := W(\widetilde{\mathbb{E}}^+)$, where, for a perfect ring R, $W(R)$ is the Witt ring of R. We put $\widetilde{\mathbb{B}}^+ := \widetilde{\mathbb{A}}^+[\frac{1}{p}]$. These rings are equipped with the weak topology and also have natural continuous G_K-actions and Frobenius actions φ. Then we have a G_K-equivariant surjection $\theta : \widetilde{\mathbb{A}}^+ \rightarrow \mathcal{O}_{\mathbb{C}_p} : \sum_{k=0}^{\infty} p^k [x_k] \mapsto \sum_{k=0}^{\infty} p^k x_k^{(0)}$, where $[\ ] : \widetilde{\mathbb{E}}^+ \rightarrow \widetilde{\mathbb{A}}^+$ is the Teichmüller lifting. By inverting p, we obtain a surjection $\widetilde{\mathbb{B}}^+ \rightarrow \mathbb{C}_p$. We put $B^+_{\mathrm{dR}} := \varprojlim_n \widetilde{\mathbb{B}}^+/(\mathrm{Ker}(\theta))^n$, which is a complete discrete valuation ring with residue field $\mathbb{C}_p$ and is equipped with the projective limit topology of the $\mathbb{Q}_p$-Banach spaces $\widetilde{\mathbb{B}}^+/(\mathrm{Ker}(\theta))^n$ whose $\mathbb{Z}_p$-lattice is the image of the natural map $\tilde{\mathbb{A}}^+ \rightarrow \tilde{\mathbb{B}}^+/(\mathrm{Ker}(\theta))^n$. Let $A_{\max}$ be the p-adic completion of $\widetilde{\mathbb{A}}^+[\frac{[\tilde{p}]}{p}]$, where $\tilde{p} := (p^{(n)})$ is an element in $\widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p, (p^{(n+1)})^p = p^{(n)}$. We put $B^+_{\max} := A_{\max}[\frac{1}{p}]$. $A_{\max}$ and $B^+_{\max}$ have continuous G_K-actions and Frobenius actions φ. We have a natural G_K-equivariant embedding $K \otimes_{K_0} B^+_{\max} \hookrightarrow B^+_{\mathrm{dR}}$. If we put $t := \log([\varepsilon]) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n}$, then we can see that $t \in A_{\max}$, $\varphi(t) = pt, g(t) = \chi_p(g)t$ for any $g \in G_K$ and $\mathrm{Ker}(\theta) = tB^+_{\mathrm{dR}} \subset B^+_{\mathrm{dR}}$ is the maximal ideal of B^+_{dR}. If we put $B_{\max} := B^+_{\max}[\frac{1}{t}], B_{\mathrm{dR}} := B^+_{\mathrm{dR}}[\frac{1}{t}]$, we have a natural embedding $K \otimes_{K_0} B_{\max} \hookrightarrow B_{\mathrm{dR}}$. We put $B_e := B^{\varphi=1}_{\max}$ on which we equipped the locally convex inductive limit topology of $B_e = \cup_n (\frac{1}{t^n} B^+_{\max})^{\varphi=1}$, where the topology on each $(\frac{1}{t^n} B^+_{\max})^{\varphi=1} = \frac{1}{t^n} B^{+\varphi=p^n}_{\max}$ is induced that of $B^+_{\max}$. We put $\mathrm{Fil}^i B_{\mathrm{dR}} := t^i B^+_{\mathrm{dR}}$ for any $i \in \mathbb{Z}$. On B_{dR}, we also equipped with the locally convex inductive limit topology of $B_{\mathrm{dR}} = \varinjlim_n \frac{1}{t^n} B^+_{\mathrm{dR}}$.

In this paper, we fix a coefficient field of p-adic representations or B-pairs of G_K. Hence we start by recalling the definition of E-coefficient versions of p-adic representations and B-pairs.

Definition 2.1. An E-representation of G_K is a finite dimensional E-vector space V with a continuous E-linear action of G_K. We call E-representation for simplicity when there is no risk of confusion about K.

Definition 2.2. A pair $W := (W_e, W^+_{\mathrm{dR}})$ is an E-B-pair if

(1) W_e is a finite $B_e \otimes_{\mathbb{Q}_p} E$-module which is free over B_e with a continuous semi-linear G_K-action.
(2) W^+_{dR} is a G_K-stable finite sub $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-module of $B_{\mathrm{dR}} \otimes_{B_e} W_e$ which generates $B_{\mathrm{dR}} \otimes_{B_e} W_e$ as a B_{dR}-module.

We have an exact fully faithful functor $W(-)$ from the category of E-representations to the category of E-B-pairs defined by

$$W(V) := (B_e \otimes_{\mathbb{Q}_p} V, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$$

for any E-representation V, where the fully faithfulness follows from Bloch-Kato's fundamental short exact sequence,

$$0 \to \mathbb{Q}_p \to B_e \oplus B^+_{\mathrm{dR}} \to B_{\mathrm{dR}} \to 0.$$

W_e is a free $B_e \otimes_{\mathbb{Q}_p} E$-module and W^+_{dR} is a free $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-module by Lemma 1.7, 1.8 of [Na09]. We define the rank of W by

$$\mathrm{rank}(W) := \mathrm{rank}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e).$$

For E-B-pairs $W_1 := (W_{1,e}, W^+_{1,\mathrm{dR}})$ and $W_2 := (W_{2,e}, W^+_{2,\mathrm{dR}})$, we define the tensor product of W_1 and W_2 by

$$W_1 \otimes W_2 := (W_{1,e} \otimes_{B_e \otimes_{\mathbb{Q}_p} E} W_{2,e}, W^+_{1,\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} W^+_{2,\mathrm{dR}})$$

and define the dual of W_1 by

$$W_1^{\vee} := (\mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_{1,e}, B_e \otimes_{\mathbb{Q}_p} E), W^{+,\vee}_{1,\mathrm{dR}})$$

where we define

$$W^{+,\vee}_{1,\mathrm{dR}} := \{f \in \mathrm{Hom}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(B_{\mathrm{dR}} \otimes_{B_e} W_{1,e}, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E) | f(W^+_{1,\mathrm{dR}}) \subseteq B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E\})$$

(remark: there is a mistake in definition 1.9 of [Na09]). The category of E-B-pairs is not an abelian category. In particular, an inclusion $W_1 \hookrightarrow W_2$ does not have a quotient in the category of E-B-pairs in general. However we can always take the saturation

$$W_1^{\mathrm{sat}} := (W^{\mathrm{sat}}_{1.e}, W^{+,\mathrm{sat}}_{1,\mathrm{dR}})$$

such that W_1^{sat} sits in $W_1 \hookrightarrow W_1^{\mathrm{sat}} \hookrightarrow W_2$ and $W_{1,e} = W^{\mathrm{sat}}_{1,e}$ and W_2/W_1^{sat} is an E-B-pair (Lemma 1.14 of [Na09]). We say that an inclusion $W_1 \hookrightarrow W_2$ is saturated if W_2/W_1 is an E-B-pair.

Next, we recall p-adic Hodge theory for B-pairs. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an E-B-pair. We define

$$D_{\mathrm{cris}}(W) := (B_{\mathrm{cris}} \otimes_{B_e} W_e)^{G_K},\ D^L_{\mathrm{cris}}(W) := (B_{\mathrm{cris}} \otimes_{B_e} W_e)^{G_L}$$

for any finite extension L of K and define

$$D_{\mathrm{dR}}(W) := (B_{\mathrm{dR}} \otimes_{B_e} W_e)^{G_K},\ D_{\mathrm{HT}}(W) := (B_{\mathrm{HT}} \otimes_{\mathbb{C}_p} (W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}))^{G_K},$$

here $B_{\mathrm{HT}} := \mathbb{C}_p[T, T^{-1}]$ on which G_K acts by $g(aT^i) := \chi_p(g)^i g(a) T^i$ for any $g \in G_K, a \in \mathbb{C}_p, i \in \mathbb{Z}$.

Definition 2.3. We say that W is crystalline (resp. de Rham, resp. Hodge-Tate) if $\dim_{K_*} D_*(W) = [E : \mathbb{Q}_p]\mathrm{rank}(W)$ for $* = \mathrm{cris}$ (resp. $* = \mathrm{dR}$, resp. $* = \mathrm{HT}$), where $K_* = K_0$ when $* = \mathrm{cris}$ and $K_* = K$ when $* = \mathrm{dR}, \mathrm{HT}$. We say that W is potentially crystalline if $\dim_{L_0}(D^L_{\mathrm{cris}}(W)) = [E : \mathbb{Q}_p]\mathrm{rank}(W)$ for a finite extension L of K, where L_0 is the maximal unramified extension of $\mathbb{Q}_p$ in L.

Definition 2.4. Let L be a finite Galois extension of K and $G_{L/K} := \mathrm{Gal}(L/K)$. We say that D is an E-filtered $(\varphi, G_{L/K})$-module over K if

(1) D is a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module with a φ-semi-linear action φ_D and a semi-linear action of $G_{L/K}$ such that $\varphi_D : D \xrightarrow{\sim} D$ is an isomorphism and that φ_D and $G_{L/K}$-action commute, where (φ-)semi-linear means that $\varphi_D(a \otimes b \cdot x) = \varphi(a) \otimes b \cdot \varphi_D(x)$, $g(a \otimes b \cdot x) = g(a) \otimes b \cdot g(x)$ for any $a \in L_0, b \in E, x \in D, g \in G_{L/K}$.

(2) $D_L := L \otimes_{L_0} D$ has a separated and exhausted decreasing filtration $\mathrm{Fil}^i D_L$ by sub $L \otimes_{\mathbb{Q}_p} E$-modules such that the $G_{L/K}$-action on D_L preserves this filtration.

Let W be a potentially crystalline E-B-pair such that $W|_{G_L}$ is crystalline for a finite Galois extension L of K, then we define an E-filtered $(\varphi, G_{L/K})$-module's structure on $D^L_{\mathrm{cris}}(W)$ as follows. First, $D^L_{\mathrm{cris}}(W)$ has a Frobenius action induced from that on B_{cris} and has a $G_{L/K}$-action induced from those on B_{cris} and W_e. We define a filtration on $L \otimes_{L_0} D^L_{\mathrm{cris}}(W) = L \otimes_K D_{\mathrm{dR}}(W)$ by

$$\mathrm{Fil}^i(L \otimes_{L_0} D^L_{\mathrm{cris}}(W)) := (L \otimes_K D_{\mathrm{dR}}(W)) \cap t^i W^+_{\mathrm{dR}}.$$

Let $D := L_0 e$ be a rank one $\mathbb{Q}_p$-filtered $(\varphi, G_{L/K})$-module with a base e, then we define $t_N(D) := v_p(\alpha)$ where $\varphi_D(e) = \alpha \cdot e$ and define $t_H(D) := i$ such that $\mathrm{Fil}^i D_L / \mathrm{Fil}^{i+1} D_L \neq 0$. For general D of rank d, we define $t_N(D) := t_N(\wedge^d D)$, $t_H(D) := t_H(\wedge^d D)$. Then we say that D is weakly admissible if $t_N(D) = t_H(D)$ and $t_N(D') \geqq t_H(D')$ for any sub $\mathbb{Q}_p$-filtered $(\varphi, G_{L/K})$-module D' of D.

Theorem 2.5. *Let L be a finite Galois extension of K, then we have the following results.*

(1) *The functor $W \mapsto D^L_{\mathrm{cris}}(W)$ gives an equivalence of categories between the category of potentially crystalline E-B-pairs which are crystalline if restricted to G_L and the category of E-filtered $(\varphi, G_{L/K})$-modules over K.*

(2) *Restricting the above functor to E-representations, the functor $V \mapsto D^L_{\mathrm{cris}}(V)$ gives an equivalence of categories between the category of potentially crystalline E-representations which are crystalline if restricted to G_L and the category of weakly admissible E-filtered $(\varphi, G_{L/K})$-modules over K.*

Proof. See Proposition 2.3.4 and Theorem 2.3.5 of [Be09] or Theorem 1.18 of [Na09]. □

Next, we recall the definition of trianguline E-B-pairs, whose deformation theory we study in detail in this chapter.

Definition 2.6. Let W be an E-B-pair of rank n, then we say that W is split trianguline if there exists a filtration

$$\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_n = W$$

by sub E-B-pairs such that W_i is saturated in W_{i+1} and W_{i+1}/W_i is a rank one E-B-pair for any $0 \leqq i \leqq n-1$. We say that W is trianguline if $W \otimes_E E'$, the

base change of W to E', is a split trianguline E'-B-pair for a finite extension E' of E.

By this definition, to study split trianguline E-B-pairs, it is important to classify rank one E-B-pairs and calculate extension classes of rank one E-B pairs, which were studied in [Na09]. We recall some results concerning these.

Theorem 2.7. *There exists a canonical one to one correspondence $\delta \mapsto W(\delta)$ between the set of continuous homomorphisms $\delta : K^\times \to E^\times$ and the set of isomorphism classes of rank one E-B-pairs.*

Proof. See Proposition 3.1 of [Co08] for $K = \mathbb{Q}_p$ and Theorem 1.45 of [Na09] for general K. For the construction of $W(\delta)$, see §1.4 of [Na09]. $\square$

This correspondence is compatible with local class field theory, i.e. for a unitary homomorphism $\delta : K^\times \to \mathcal{O}_E^\times$ if we take the character $\tilde{\delta} : G_K^{\mathrm{ab}} \to \mathcal{O}_E^\times$ satisfying $\tilde{\delta} \circ \mathrm{rec}_K = \delta$, then we have an isomorphism

$$W(\delta) \xrightarrow{\sim} W(E(\tilde{\delta})).$$

This correspondence is also compatible with tensor products and with duals, i.e for continuous homomorphisms $\delta_1, \delta_2 : K^\times \to E^\times$, we have

$$W(\delta_1) \otimes W(\delta_2) \xrightarrow{\sim} W(\delta_1\delta_2) \text{ and } W(\delta_1)^\vee \xrightarrow{\sim} W(\delta_1^{-1}).$$

There are some important examples of rank one E-B-pairs which we recall now. For any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}$, we define a homomorphism

$$\prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma} : K^\times \to E^\times : y \mapsto \prod_{\sigma \in \mathcal{P}} \sigma(y)^{k_\sigma},$$

then we have an isomorphism

$$W(\prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}) \xrightarrow{\sim} (B_e \otimes_{\mathbb{Q}_p} E, \oplus_{\sigma \in \mathcal{P}} t^{k_\sigma} B_{\mathrm{dR}}^+ \otimes_{K,\sigma} E)$$

(Lemma 2.12 of [Na09]). Let $N_{K/\mathbb{Q}_p} : K^\times \to \mathbb{Q}_p^\times$ be the norm and $|-| : \mathbb{Q}_p^\times \to \mathbb{Q}^\times \hookrightarrow E^\times$ be the p-adic absolute value such that $|p| = \frac{1}{p}$, and we define $|N_{K/\mathbb{Q}_p}| : K^\times \to E^\times$ the composite of $N_{K/\mathbb{Q}_p}$ and $|-|$, then we have an isomorphism

$$W(|N_{K/\mathbb{Q}_p}| \prod_{\sigma \in \mathcal{P}} \sigma) \xrightarrow{\sim} W(E(\chi_p)),$$

which is the E-B-pair associated to p-adic cyclotomic character. Next, we recall the definition and some properties of Galois cohomology of E-B-pairs. For an E-B-pair $W := (W_e, W_{\mathrm{dR}}^+)$, we put $W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$. We have natural inclusions, $W_e \hookrightarrow W_{\mathrm{dR}}$, $W_{\mathrm{dR}}^+ \hookrightarrow W_{\mathrm{dR}}$. We define the Galois cohomology $\mathrm{H}^i(G_K, W)$ of W as the continuous cochains' cohomology of the complex

$$W_e \oplus W_{\mathrm{dR}}^+ \to W_{\mathrm{dR}} : (x, y) \mapsto x - y$$

(see the appendix of this article or §2.1 of [Na09] for the precise definition). As in the usual p-adic representation case, we have the following isomorphisms of E-vector spaces

$$\mathrm{H}^0(G_K, W) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(B_E, W),\ \mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(B_E, W),$$

where $B_E := (B_e \otimes_{\mathbb{Q}_p} E, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)$ is the trivial E-B-pair and $\mathrm{Hom}_{G_K}(-,-)$ is the group of homomorphisms of E-B-pairs and $\mathrm{Ext}^1(-,-)$ is the extension class group in the category of E-B-pairs. If V is an E-representation of G_K, we have a canonical isomorphism

$$\mathrm{H}^i(G_K, V) \xrightarrow{\sim} \mathrm{H}^i(G_K, W(V)),$$

which follows from Bloch-Kato's fundamental short exact sequence. Moreover, we have the following theorem, Euler-Poincaré characteristic formula and Tate duality theorem for B-pairs.

Theorem 2.8. *Let W be an E-B-pair.*

(1) *For $i = 0, 1, 2$, $\mathrm{H}^i(G_K, W)$ is finite dimensional over E and $\mathrm{H}^i(G_K, W) = 0$ for $i \neq 0, 1, 2$.*
(2) $\sum_{i=0}^{2}(-1)^{i-1}\dim_E \mathrm{H}^i(G_K, W) = [K : \mathbb{Q}_p]\mathrm{rank}(W)$.
(3) *Let W be a $\mathbb{Q}_p$-B-pair. For any $i = 0, 1, 2$, there is a natural perfect pairing defined by cup product,*

$$\begin{aligned}\mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) &\to \mathrm{H}^2(G_K, W \otimes W^\vee(\chi_p)) \\ &\to \mathrm{H}^2(G_K, W(\mathbb{Q}_p(\chi_p)) \xrightarrow{\sim} \mathbb{Q}_p,\end{aligned}$$

where the last isomorphism is Tate's trace map.

Proof. See Theorem 5.9 and and Theorem 5.10 in the appendix. □

Using these formulae, we obtain the following dimension formulae of Galois cohomologies of rank one E-B-pairs.

Proposition 2.9. *Let $\delta : K^\times \to E^\times$ be a continuous homomorphism, then we have*:

(1) $\mathrm{H}^0(G_K, W(\delta)) \xrightarrow{\sim} E$ *if* $\delta = \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$, *and* $\mathrm{H}^0(G_K, W(\delta)) = 0$ *otherwise.*
(2) $\mathrm{H}^2(G_K, W(\delta)) \xrightarrow{\sim} E$ *if* $\delta = |N_{K/\mathbb{Q}_p}| \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 1}$, *and* $\mathrm{H}^2(G_K, W(\delta)) = 0$ *otherwise.*
(3) $\dim_E \mathrm{H}^1(G_K, W(\delta)) = [K : \mathbb{Q}_p] + 1$ *if* $\delta = \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$ *or* $\delta = |N_{K/\mathbb{Q}_p}| \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 1}$, *and* $\dim_E \mathrm{H}^1(G_K, W(\delta)) = [K : \mathbb{Q}_p]$ *otherwise.*

Proof. See Theorem 2.9 and Theorem 2.22 of [Co08] for $K = \mathbb{Q}_p$. For general K, the results can be proved by using Proposition 2.14 and Proposition 2.15 of [Na09] and Tate duality for B-pairs. □

2.1.2. *B-pairs over Artin local rings.* Here, we define B-pairs with Artin ring coefficients, which we need to define the notion of deformations of E-B-pairs. Let $\mathcal{C}_E$ be the category of Artin local E-algebra A with the residue field E. The morphisms in $\mathcal{C}_E$ are given by local E-algebra homomorphisms. For any $A \in \mathcal{C}_E$, we denote by $\mathfrak{m}_A$ the maximal ideal of A. We define the A-coefficient version of B-pairs as follows.

Definition 2.10. We call a pair $W := (W_e, W^+_{\mathrm{dR}})$ an A-B-pair of G_K if

(1) W_e is a finite $B_e \otimes_{\mathbb{Q}_p} A$-module which is flat over A and is free over B_e, with a continuous semi-linear G_K-action.
(2) W^+_{dR} is a finite generated sub $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module of $B_{\mathrm{dR}} \otimes_{B_e} W_e$ which is stable by the G_K-action and which generates $B_{\mathrm{dR}} \otimes_{B_e} W_e$ as a $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module such that $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is flat over A.

For an A-B-pair $W := (W_e, W^+_{\mathrm{dR}})$, we put $W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$.

We simply call an A-B-pair if there is no risk of confusing about K.

Lemma 2.11. *Let $W := (W_e, W^+_{\mathrm{dR}})$ be an A-B-pair. Then W_e is a finite free $B_e \otimes_{\mathbb{Q}_p} A$-module, W^+_{dR} is a finite free $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module and $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is a finite free $\mathbb{C}_p \otimes_{\mathbb{Q}_p} A$-module.*

Proof. First, we prove for W_e. Because the sub module $\mathfrak{m}_A W_e \subseteq W_e$ is a G_K-stable finite generated torsion free B_e-module and because B_e is a Bézout domain by Proposition 1.1.9 of [Be08], $\mathfrak{m}_A W_e$ is a finite free B_e-module by Lemma 2.4 of [Ke04]. By Lemma 2.1.4 of [Be08], the cokernel $W_e \otimes_A E$ is also a finite free B_e-module (with an E-action). By Lemma 1.7 of [Na09], $W_e \otimes_A E$ is a finite free $B_e \otimes_{\mathbb{Q}_p} E$-module of some rank n. We take a $B_e \otimes_{\mathbb{Q}_p} A$-linear morphism $f : (B_e \otimes_{\mathbb{Q}_p} A)^n \to W_e$ which is a lift of a $B_e \otimes_{\mathbb{Q}_p} E$-linear isomorphism $(B_e \otimes_{\mathbb{Q}_p} E)^n \xrightarrow{\sim} W_e \otimes_A E$. Because A is Artinian, then f is surjective. Because W_e is A-flat, we have $\mathrm{Ker}(f) \otimes_A E = 0$, hence $\mathrm{Ker}(f) = 0$. Hence W_e is a free $B_e \otimes_{\mathbb{Q}_p} A$-module.

Next, we prove that W^+_{dR} is a free $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module. Because W_e is a free $B_e \otimes_{\mathbb{Q}_p} A$-module, W_{dR} is a free $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module, in particular this is flat over A. Because $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is a flat A-module, $W_{\mathrm{dR}}/W^+_{\mathrm{dR}}$ is also a flat A-module. Hence W^+_{dR} is also flat over A. By the A-flatness of $W_{\mathrm{dR}}/W^+_{\mathrm{dR}}$, we have an inclusion $W^+_{\mathrm{dR}} \otimes_A E \hookrightarrow W_{\mathrm{dR}} \otimes_A E$, hence $W^+_{\mathrm{dR}} \otimes_A E$ is a finite generated torsion free B^+_{dR}-module, hence free B^+_{dR}-module. By Lemma 1.8 of [Na09], we can show that W^+_{dR} is a free $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module in the same way as in the case of W_e. $\mathbb{C}_p \otimes_{\mathbb{Q}_p} A$-freeness of $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ follows from $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-freeness of W^+_{dR}.

□

Definition 2.12. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an A-B-pair. We define the rank of W by $\mathrm{rank}(W) := \mathrm{rank}_{B_e \otimes_{\mathbb{Q}_p} A}(W_e)$.

Definition 2.13. Let $f : A \to A'$ be a morphism in $\mathcal{C}_E$ and $W = (W_e, W_{\mathrm{dR}})$ be an A-B-pair. We define the base change of W to A' by

$$W \otimes_A A' := (W_e \otimes_A A', W^+_{\mathrm{dR}} \otimes_A A').$$

By Lemma 2.11, we can easily see that this is an A'-B-pair.

Definition 2.14. Let $W_1 = (W_{e,1}, W^+_{\mathrm{dR},1})$, $W_2 = (W_{e,2}, W^+_{\mathrm{dR},2})$ be A-B-pairs. We define the tensor product of W_1 and W_2 by $W_1 \otimes W_2 := (W_{e,1} \otimes_{B_e \otimes_{\mathbb{Q}_p} A} W_{e,2}, W^+_{\mathrm{dR},1} \otimes_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A} W^+_{\mathrm{dR},2})$, and define the dual of W_1 by $W_1^{\vee} := (\mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} A}(W_{e,1}, B_e \otimes_{\mathbb{Q}_p} A), W^{+,\vee}_{\mathrm{dR},1})$. Here, $W^{+,\vee}_{\mathrm{dR},1} := \{f \in \mathrm{Hom}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A}(W_{\mathrm{dR},1}, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A) | f(W^+_{\mathrm{dR},1}) \subseteq B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A\}$. By Lemma 2.11, we can easily see that these are A-B-pairs.

Next, we classify rank one A-B-pairs. Let $\delta : K^{\times} \to A^{\times}$ be a continuous homomorphism, then we define a rank one A-B-pair $W(\delta)$ as follows. Let $\bar{u} \in E^{\times}$ be the reduction of $u := \delta(\pi_K)$. We define a homomorphism $\delta_0 : K^{\times} \to A^{\times}$ such that $\delta_0|_{\mathcal{O}_K^{\times}} = \delta|_{\mathcal{O}_K^{\times}}$, $\delta_0(\pi_K) = u/\bar{u}$. Because $u/\bar{u} \in 1 + \mathfrak{m}_A$, $(u/\bar{u})^{p^n}$ converges to $1 \in A^{\times}$. If we fix an isomorphism $K^{\times} \xrightarrow{\sim} \mathcal{O}_K^{\times} \times \mathbb{Z} : v\pi_K^n \mapsto (v, n)$ $(v \in \mathcal{O}_K^{\times})$, then δ_0 uniquely extends to a continuous homomorphism $\delta'_0 : \mathcal{O}_K^{\times} \times \hat{\mathbb{Z}} \to \mathcal{O}_K^{\times} \times \mathbb{Z}_p \to A^{\times}$. By local class field theory, then there exists unique character $\tilde{\delta}_0 : G_K^{\mathrm{ab}} \to A^{\times}$ such that $\delta_0 = \tilde{\delta}_0 \circ \mathrm{rec}_K$, where $\mathrm{rec}_K : K^{\times} \to G_K^{\mathrm{ab}}$ is the reciprocity map as in Notation. Using $\tilde{\delta}_0$, we define an étale rank one A-B-pair $W(A(\tilde{\delta}_0))$, which is the A-B-pair associated to the rank one A-representation $A(\tilde{\delta}_0)$. Next, we define a non-étale rank one A-B-pair by using $\bar{u} \in E^{\times}$. For this, first we define a rank one E-filtered φ-module $D_{\bar{u}} := K_0 \otimes_{\mathbb{Q}_p} Ee_{\bar{u}}$ such that $\varphi^f(e_{\bar{u}}) := \bar{u}e_{\bar{u}}$ and $\mathrm{Fil}^0(K \otimes_{K_0} D_{\bar{u}}) := K \otimes_{K_0} D_{\bar{u}}$, $\mathrm{Fil}^1(K \otimes_{K_0} D_{\bar{u}}) := 0$. From this, we obtain the rank one crystalline E-B-pair $W(D_{\bar{u}})$ such that $D_{\mathrm{cris}}(W(D_{\bar{u}})) \xrightarrow{\sim} D_{\bar{u}}$ which is pure of slope $\frac{v_p(\bar{u})}{f}$. By tensoring these, we define a rank one A-B-pair $W(\delta)$ by $W(\delta) := (W(D_{\bar{u}}) \otimes_E A) \otimes W(A(\tilde{\delta}_0))$, which is pure of slope $\frac{v_p(\bar{u})}{f}$.

The following proposition is the A-coefficient version of Theorem 1.45 of [Na09].

Proposition 2.15. *This construction $\delta \mapsto W(\delta)$ does not depend on the choice of uniformizer π_K and gives a bijection between the set of continuous homomorphisms $\delta : K^{\times} \to A^{\times}$ and the set of isomorphism classes of rank one A-B-pairs.*

Proof. The independence of the choice of uniformizer and the injection is proved in the same way as in the proof of Theorem 1.45 of [Na09]. We prove the surjection. Let W be a rank one A-B-pair, then as an E-B-pair, W is a successive extension of rank one E-B-pair $W \otimes_A E$. $W \otimes_A E$ is pure of slope $\frac{n}{fe_E}$ for some $n \in \mathbb{Z}$ by Lemma 1.42 of [Na09]. Hence, by Theorem 1.6.6 of [Ke08], W is also pure of slope $\frac{n}{fe_E}$. We define a rank one E-filtered φ-module $D_{\pi_E^n} := K_0 \otimes_{\mathbb{Q}_p} Ee_{\pi_E^n}$ in the same way as in $D_{\bar{u}}$, where π_E is a uniformizer of E, then $W \otimes (W(D_{\pi_E^n}) \otimes_E A)^{\vee}$ is pure of slope zero by Lemma 1.34 of [Na09]. Hence, there exists $\tilde{\delta}' : G_K^{\mathrm{ab}} \to A^{\times}$ such

that $W \otimes (W(D_{\pi_E^n}) \otimes_E A)^\vee \xrightarrow{\sim} W(A(\tilde{\delta}'))$. We put $\delta' := \tilde{\delta}' \circ \mathrm{rec}_K : K^\times \to A^\times$ and define $\delta : K^\times \to A^\times$ such that $\delta|_{\mathcal{O}_K^\times} := \delta'|_{\mathcal{O}_K^\times}$ and $\delta(\pi_K) := \delta'(\pi_K)\pi_E^n$, then we have an isomorphism $W \xrightarrow{\sim} W(\delta)$, which can be easily seen from the construction of $W(\delta)$.

□

By local class field theory, we have a canonical bijection $\delta \mapsto A(\tilde{\delta})$ from the set of unitary continuous homomorphisms from $K^\times$ to $A^\times$ (where unitary means that the reduction of the image of δ is contained in $\mathcal{O}_E^\times$) to the set of isomorphism class of rank one A-representations of G_K, where $\tilde{\delta} : G_K^{ab} \to A^\times$ is the continuous homomorphism such that $\delta = \tilde{\delta} \circ \mathrm{rec}_K$. By the definition of $W(\delta)$ and by the above proof, it is easy to see that there exists an isomorphism $W(\delta) \xrightarrow{\sim} W(A(\tilde{\delta}))$ for any unitary homomorphism $\delta : K^\times \to A^\times$. Moreover, it is easy to see that for any continuous homomorphisms $\delta_1, \delta_2 : K^\times \to A^\times$ we have isomorphisms $W(\delta_1) \otimes W(\delta_2) \xrightarrow{\sim} W(\delta_1\delta_2)$ and $W(\delta_1)^\vee \xrightarrow{\sim} W(\delta_1^{-1})$.

Next, we generalize the functor D_{cris} to potentially crystalline A-B-pairs. First, we define the A-coefficient version of filtered (φ, G_K)-modules. Let L be a finite Galois extension of K, we denote by $G_{L/K} := \mathrm{Gal}(L/K)$.

Definition 2.16. Let A be an object of $\mathcal{C}_E$. We say that D is an A-filtered $(\varphi, G_{L/K})$-module of K if D satisfies the following conditions.

(1) D is a finite $L_0 \otimes_{\mathbb{Q}_p} A$-module which is free as an A-module with a φ-semi-linear action $\varphi : D \xrightarrow{\sim} D$.

(2) $D_L := L \otimes_{L_0} D$ has a decreasing filtration $\mathrm{Fil}^i D_L$ by sub $L \otimes_{\mathbb{Q}_p} A$-modules such that $\mathrm{Fil}^k D_L = 0$ and $\mathrm{Fil}^{-k} D_L = D_L$ for sufficiently large k and that $\mathrm{Fil}^k D_L / \mathrm{Fil}^{k+1} D_L$ are free A-modules for any k.

(3) $G_{L/K}$ acts on D by $L_0 \otimes_{\mathbb{Q}_p} A$-semi-linear automorphism which commutes with the action of φ and preserves the filtration.

Remark 2.17. Using the φ-structure on D, we can see that D is a free $L_0 \otimes_{\mathbb{Q}_p} A$-module.

Let $W_A := (W_{A,e}, W_{A,\mathrm{dR}}^+)$ be an A-B-pair such that $W_A|_{G_L}$ is crystalline (as an E-B-pair) for a finite Galois extension L of K. As in the case of E-B-pairs, we define $D_{\mathrm{cris}}^L(W_A) := (B_{\mathrm{cris}} \otimes_{B_e} W_e)^{G_L}$ with a φ-action induced from that on B_{cris}, then the natural map $L \otimes_{L_0} D_{\mathrm{cris}}^L(W_A) \to D_{\mathrm{dR}}^L(W_A) := (B_{\mathrm{dR}} \otimes_{B_e} W_e)^{G_L}$ is isomorphism. We define $\mathrm{Fil}^k D_{\mathrm{dR}}^L(W_A) := D_{\mathrm{dR}}^L(W_A) \cap t^k W_{\mathrm{dR}}^+$ for any $k \in \mathbb{Z}$. These are equipped with a $G_{L/K}$-action naturally.

Lemma 2.18. *In the above situation, $D_{\mathrm{cris}}^L(W_A)$ is an A-filtered $(\varphi, G_{L/K})$-module of K.*

Proof. It suffices only to show the A-freeness of $D_{\mathrm{cris}}^L(W_A)$, $\mathrm{Fil}^k D_{\mathrm{dR}}^L(W_A)$, $\mathrm{Fil}^k D_{\mathrm{dR}}^L(W_A)/\mathrm{Fil}^{k+1} D_{\mathrm{dR}}^L(W_A)$. Here, we only prove the A-freeness of $D_{\mathrm{cris}}^L(W_A)$, other cases

can be proved in a similar way. By the exactness of D^L_{cris} for E-B-pairs which are crystalline when restricted to G_L, we have a natural isomorphism $D^L_{\mathrm{cris}}(W_A)\otimes_A N\xrightarrow{\sim} D^L_{\mathrm{cris}}(W_A\otimes_A N)$ for any finite A-module N. From this, for any A-linear injection $N_1\hookrightarrow N_2$ of finite A-modules, we have an inclusion $D^L_{\mathrm{cris}}(W_A)\otimes_A N_1 = D^L_{\mathrm{cris}}(W_A\otimes_A N_1)\hookrightarrow D^L_{\mathrm{cris}}(W_A\otimes_A N_2) = D^L_{\mathrm{cris}}(W_A)\otimes_A N_2$ because $W_{A,e}$ is A-flat. Hence, $D^L_{\mathrm{cris}}(W_A)$ is A-flat. □

Conversely, let D be an A-filtered $(\varphi, G_{L/K})$-module of K, then we define $W_e(D) := (B_{\mathrm{cris}}\otimes_{L_0} D)^{\varphi=1}$. We have a natural isomorphism $B_{\mathrm{dR}}\otimes_{B_e} W_e(D)\xrightarrow{\sim} B_{\mathrm{dR}}\otimes_L D_L$. We define $W^+_{\mathrm{dR}}(D) := \mathrm{Fil}^0(B_{\mathrm{dR}}\otimes_L D_L)\subseteq B_{\mathrm{dR}}\otimes_{B_e} W_e(D)$. We write $W(D) := (W_e(D), W^+_{\mathrm{dR}}(D))$ which is a potentially crystalline E-B-pair with an A-action.

Lemma 2.19. *In the above situation, $W(D)$ is a potentially crystalline A-B-pair.*

Proof. It suffices to show the A-flatness of $W_e(D)$ and $W^+_{\mathrm{dR}}(D)/tW^+_{\mathrm{dR}}(D)$. We can prove these in the same way as in Lemma 2.18 by using the exactness of the functor $W(D)$ and the A-flatness of D and $\mathrm{Fil}^kD_L/\mathrm{Fil}^{k+1}D_L$ for any k. □

Corollary 2.20. *Let A be an object in $\mathcal{C}_E$. The functor D^L_{cris} gives an equivalence of categories between the category of potentially crystalline A-B-pairs which are crystalline if restricted to G_L and the category of A-filtered $(\varphi, G_{L/K})$-modules of K.*

Proof. This follows from Lemma 2.18 and Lemma 2.19 and Theorem 2.5. □

Next, we prove some lemmas which will be used in later sections.

Let W_A be an A-B-pair which is not potentially crystalline in general and L be a finite Galois extension of K, then we can define $D^L_{\mathrm{cris}}(W_A)$ in the same way as in the case where W_A is potentially crystalline. This is an E-filtered $(\varphi, G_{L/K})$-module, but in general this may not be an A-filtered $(\varphi, G_{L/K})$-module.

Lemma 2.21. *Let W be an E-B-pair. Let $D\subseteq D_{\mathrm{cris}}(W)$ be a rank one sub E-filtered φ-module whose filtration is induced from that of $D_{\mathrm{cris}}(W)$, then there exists a natural saturated inclusion $W(D)\hookrightarrow W$.*

Proof. Twisting W by a suitable crystalline character of the form $\prod_{\sigma\in\mathcal{P}}\sigma(\chi_{\mathrm{LT}})^{k_\sigma}$, we may assume that $\mathrm{Fil}^0(D_K) = D_K$ and $\mathrm{Fil}^1(D_K) = 0$, where we put $D_K := K\otimes_{K_0} D$. We have natural inclusions $W(D)_e = (B_{\mathrm{cris}}\otimes_{K_0} D)^{\varphi=1}\subseteq (B_{\mathrm{cris}}\otimes_{K_0} D_{\mathrm{cris}}(W))^{\varphi=1}\subseteq (B_{\mathrm{cris}}\otimes_{B_e} W_e)^{\varphi=1} = W_e$ and, under the above assumption, $W^+_{\mathrm{dR}}(D) = \mathrm{Fil}^0(B_{\mathrm{dR}}\otimes_K D_K) = B^+_{\mathrm{dR}}\otimes_K D_K\subseteq \mathrm{Fil}^0(B_{\mathrm{dR}}\otimes_K D_{\mathrm{dR}}(W))\subseteq W^+_{\mathrm{dR}}$, which define an inclusion $W(D)\hookrightarrow W$. Hence, it suffices to show that this inclusion is saturated, i.e. $B^+_{\mathrm{dR}}\otimes_K D_K = (B_{\mathrm{dR}}\otimes_K D_K)\cap W^+_{\mathrm{dR}}$. We can write $(B_{\mathrm{dR}}\otimes_K D_K)\cap W^+_{\mathrm{dR}} = \oplus_{\sigma\in\mathcal{P}}\frac{1}{t^{k_\sigma}}B^+_{\mathrm{dR}}\otimes_{K,\sigma} D_{K,\sigma}$ for some $k_\sigma\in\mathbb{Z}_{\geqq 0}$, where we decompose D_K by $D_K\xrightarrow{\sim}\oplus_{\sigma\in\mathcal{P}}D_K\otimes_{K\otimes_{\mathbb{Q}_p}E,\sigma\otimes id_E} E =: \oplus_{\sigma\in\mathcal{P}}D_{K,\sigma}$. If $k_\sigma\geqq 1$ for a $\sigma\in\mathcal{P}$, then $D_{K,\sigma}\subseteq t^{k_\sigma}W^+_{\mathrm{dR}}$. Because the filtration on D is induced from $D_{\mathrm{cris}}(W)$, this implies

that $\mathrm{Fil}^{k_\sigma} D_{K,\sigma} = D_{K,\sigma}$, this contradicts to $\mathrm{Fil}^1 D_{K,\sigma} = 0$. Hence $k_\sigma = 0$ for any $\sigma \in \mathcal{P}$, this implies that $B^+_{\mathrm{dR}} \otimes_K D_K = (B_{\mathrm{dR}} \otimes_K D_K) \cap W^+_{\mathrm{dR}}$.

$\square$

Lemma 2.22. *Let W_A be an A-B-pair. Let $D \subseteq D_{\mathrm{cris}}(W_A)$ be a sub E-filtered φ-module which is an A-filtered φ-module of rank one, where the filtration on D is the one induced from that of $D_{\mathrm{cris}}(W_A)$. We assume that the natural map $D \otimes_A E \to D_{\mathrm{cris}}(W_A \otimes_A E)$ remains injective. Then, we have a natural injection of A-B-pairs $W(D) \hookrightarrow W_A$ such that the cokernel $W_A/W(D)$ is also an A-B-pair.*

Proof. In the same way as in the above proof, we have a natural injection $W(D) \hookrightarrow W_A$. Because the natural map $D \otimes_A E \to D_{\mathrm{cris}}(W_A \otimes_A E)$ is injection, we obtain an injection $W(D) \otimes_A E \xrightarrow{\sim} W(D \otimes_A E) \hookrightarrow W_A \otimes_A E$ and this injection is saturated by the above lemma. Hence, it suffices to show that if $W_1 \hookrightarrow W_2$ is an inclusion of A-B-pairs such that $W_1 \otimes_A E \to W_2 \otimes_A E$ remains injection and saturated, then the cokernel W_2/W_1 exists and is an A-B-pair. We put $W_{3,e}$, $W^+_{3,\mathrm{dR}}$ the cokernels of $W_{1,e} \hookrightarrow W_{2,e}$, $W^+_{1,\mathrm{dR}} \hookrightarrow W^+_{2,\mathrm{dR}}$ respectively. By Lemma 2.2.3 (i) of [Bel-Ch09], these are A-flat. Hence, it suffices to show that these are free over B_e, B^+_{dR} respectively. We can prove this claim in the same way as in Lemma 2.2.3 (iii) of [Bel-Ch09].

$\square$

2.2. **Deformations of B-pairs.** In this subsection, we develop the deformation theory of B-pairs, which are the generalization of Mazur's (characteristic zero) deformation theory of p-adic Galois representations.

Definition 2.23. Let A be an object in $\mathcal{C}_E$, W be an E-B-pair. We say that a pair (W_A, ι) is a deformation of W over A if W_A is an A-B-pair and $\iota : W_A \otimes_A E \xrightarrow{\sim} W$ is an isomorphism of E-B-pairs. Let (W_A, ι), (W'_A, ι') be two deformations of W over A. Then we say that (W_A, ι) and (W'_A, ι') are equivalent if there exists an isomorphism $f : W_A \xrightarrow{\sim} W'_A$ of A-B-pairs which satisfies $\iota = \iota' \circ \bar{f}$, where $\bar{f} : W_A \otimes_A E \xrightarrow{\sim} W'_A \otimes_A E$ is the reduction of f.

Definition 2.24. Let W be an E-B-pair, then we define the deformation functor D_W from the category $\mathcal{C}_E$ to the category of sets by

$$D_W(A) := \{ \text{ equivalent classes } (W_A, \iota) \text{ of deformations of } W \text{ over } A\}$$

for any $A \in \mathcal{C}_E$.

We simply denote by W_A if there is no risk of confusing about ι.

Next, we prove the pro-representability of the functor D_W under suitable conditions. For this, we recall Schlessinger's criterion for pro-representability of functors from $\mathcal{C}_E$ to the category of sets. We call a morphism $f : A' \to A$ in $\mathcal{C}_E$ a small extension if it is surjective and the kernel $\mathrm{Ker}(f) = (t)$ is generated by a nonzero single element $t \in A'$ and $\mathrm{Ker}(f) \cdot \mathfrak{m}_{A'} = 0$. $E[\varepsilon]$ is the ring defined by $E[\varepsilon] := E[X]/(X^2)$.

Theorem 2.25. *Let F be a functor from $\mathcal{C}_E$ to the category of sets such that $F(E)$ is a single point. For morphisms $A' \to A$, $A" \to A$ in $\mathcal{C}_E$, consider the natural map*

(1) $F(A' \times_A A") \to F(A') \times_{F(A)} F(A")$,

then F is pro-representable if and only if F satisfies properties (H_1), (H_2), (H_3), (H_4) below:

(H_1) (1) *is surjective if $A" \to A$ is surjective.*

(H_2) (1) *is bijective when $A = E$ and $A" = E[\varepsilon]$.*

(H_3) $\dim_E(t_F) < \infty$ (*where $t_F := F(E[\varepsilon])$ and, under the condition (H_2), it is known that t_F has a natural E-vector space structure*).

(H_4) (1) *is bijective if $A' = A"$ and $A' \to A$ is a small extension.*

Proof. See [Schl68] or §18 of [Ma97]. □

Using this criterion, we prove the pro-representability of D_W.

Proposition 2.26. *Let W be an E-B-pair. If $\mathrm{End}_{G_K}(W) = E$, then D_W is pro-representable by a complete noetherian local E-algebra R_W with residue field E.*

For proving this proposition, we first prove some lemmas.

Lemma 2.27. *Let $\mathrm{ad}(W) := \mathrm{Hom}(W, W)(\xrightarrow{\sim} W \otimes W^{\vee})$ be the internal endomorphism of W, then there exists an isomorphism of E-vector spaces $D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W))$.*

Proof. Let $W_{E[\varepsilon]} := (W_{E[\varepsilon],e}, W^+_{E[\varepsilon],\mathrm{dR}})$ be a deformation of W over $E[\varepsilon]$. From this, we define an element in $\mathrm{H}^1(G_K, \mathrm{ad}(W))$ as follows. Because $\varepsilon W_{E[\varepsilon],e} \xrightarrow{\sim} W_e$ and $W_{E[\varepsilon],e}/\varepsilon W_{E[\varepsilon],e} \xrightarrow{\sim} W_e$ (where we put $W := (W_e, W^+_{\mathrm{dR}})$), we have a natural exact sequence of $B_e \otimes_{\mathbb{Q}_p} E[G_K]$-modules

$$0 \to W_e \to W_{E[\varepsilon],e} \to W_e \to 0$$

We fix an isomorphism of $B_e \otimes_{\mathbb{Q}_p} E$-modules $W_{E[\varepsilon],e} \xrightarrow{\sim} W_e e_1 \oplus W_e e_2$ such that first factor $W_e e_1$is equal to $\varepsilon W_{E[\varepsilon]}$ as $B_e \otimes_{\mathbb{Q}_p} E[G_K]$-module and that the above natural projection maps the second factor $W_e e_2$ to W_e by $x e_2 \mapsto x$ for any $x \in W_e$. We define a continuous one cocycle by

$$c_e : G_K \to \mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e) \text{ by } g(y e_2) := c_e(g)(gy)e_1 + gye_2$$

for any $g \in G_K$ and $y \in W_e$. For W^+_{dR}, we fix an isomorphism $W^+_{E[\varepsilon],\mathrm{dR}} \xrightarrow{\sim} W^+_{\mathrm{dR}} e_1 \oplus W^+_{\mathrm{dR}} e'_2$ as in the case of W_e, then we define a one cocycle by

$$c_{\mathrm{dR}} : G_K \to \mathrm{Hom}_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W^+_{\mathrm{dR}}, W^+_{\mathrm{dR}}) \text{ by } g(y e'_2) := c_{\mathrm{dR}}(g)(gy)e_1 + gye'_2$$

for any $g \in G_K$ and for any $y \in W^+_{\mathrm{dR}}$. Next, we define $c \in \mathrm{Hom}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$ as follows. Tensoring $W_{E[\varepsilon],e}$ and $W^+_{E[\varepsilon],\mathrm{dR}}$ with B_{dR} over B_e or B^+_{dR}, we have an

isomorphism $f : W_{\mathrm{dR}}e_1 \oplus W_{\mathrm{dR}}e_2 \xrightarrow{\sim} W_{E[\varepsilon],\mathrm{dR}} \xrightarrow{\sim} W_{\mathrm{dR}}e_1 \oplus W_{\mathrm{dR}}e'_2$ of $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-modules. We define by

$$c : W_{\mathrm{dR}} \to W_{\mathrm{dR}} \text{ by } f(ye_2) := c(y)e_1 + ye'_2$$

for any $y \in W_{\mathrm{dR}}$. By the definition, the triple $(c_e, c_{\mathrm{dR}}, c)$ satisfies

$$c_e(g) - c_{\mathrm{dR}}(g) = gc - c \text{ in } \mathrm{Hom}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$$

for any $g \in G_K$, i.e. the triple $(c_e, c_{\mathrm{dR}}, c)$ defines an element of $\mathrm{H}^1(G_K, \mathrm{ad}(W))$ by the definition of Galois cohomology of B-pairs (§ 2.1 of [Na09]), then it is standard to check that this definition is independent of the choice of fixed isomorphism $W_{E[\varepsilon],e} \xrightarrow{\sim} W_e e_1 \oplus W_e e_2$, etc, and it is easy to check that this map defines an isomorphism $D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W))$.

□

Lemma 2.28. *Let W_A be a deformation of W over A. If $\mathrm{End}_{G_K}(W) = E$, then $\mathrm{End}_{G_K}(W_A) = A$.*

Proof. We prove this lemma by induction on the length of A. When $A = E$, this is trivial. We assume that this lemma is proved for the rings of length n and assume that A is of length $n + 1$. We take a small extension $f : A \to A'$. Because $\mathrm{End}_{G_K}(W) = \mathrm{H}^0(G_K, W^\vee \otimes W)$, then we have the following short exact sequence,

$$0 \to \mathrm{Ker}(f) \otimes_E \mathrm{End}_{G_K}(W) \to \mathrm{End}_{G_K}(W_A) \to \mathrm{End}_{G_K}(W_A \otimes_A A').$$

From this and the induction hypothesis, we have

$$\begin{aligned} \mathrm{length}(\mathrm{End}_{G_K}(W_A)) &\leqq \mathrm{length}(\mathrm{End}_{G_K}(W_A \otimes_A A')) + \mathrm{length}(\mathrm{Ker}(f) \otimes_E \mathrm{End}_{G_K}(W)) \\ &= \mathrm{length}(A') + 1 = \mathrm{length}(A). \end{aligned}$$

On the other hand, we have a natural inclusion $A \subseteq \mathrm{End}_{G_K}(W_A)$. Comparing length, we have an equality $A = \mathrm{End}_{G_K}(W_A)$. □

Proof. (of proposition) Let W be a rank n E-B-pair satisfying $\mathrm{End}_{G_K}(W) = E$. For this W, we check the conditions (H_i) of Schlessinger's criterion. First, by Lemma 2.27, we have

$$\dim_E(D_W(E[\varepsilon])) = \dim_E(\mathrm{H}^1(G_K, \mathrm{ad}(W))) < \infty,$$

hence (H_3) is satisfied. Next we check (H_1). Let $f : A' \to A$, $g : A" \to A$ be morphisms in $\mathcal{C}_E$ such that g is surjective. Let $([W_{A'}], [W_{A"}])$ be an element in $D_W(A') \times_{D_W(A)} D_W(A")$. We take deformations $W_{A'} := (W_{A',e}, W^+_{A',\mathrm{dR}})$, $W_{A"} := (W_{A",e}, W^+_{A",\mathrm{dR}})$ over A' and $A"$ which are contained in equivalent classes $[W_{A'}]$ and $[W_{A"}]$ respectively, then we have an isomorphism $h : W_{A'} \otimes_{A'} A \xrightarrow{\sim} W_{A"} \otimes_{A"} A =: W_A := (W_{A,e}, W^+_{A,\mathrm{dR}})$ which defines an equivalent class in $D_W(A)$. We fix a basis $e_1, \cdots, e_n$ of $W_{A',e}$ as a $B_e \otimes_{\mathbb{Q}_p} A'$ -module and denote by $\overline{e}_1, \cdots, \overline{e}_n$ the basis of $W_{A',e} \otimes_{A'} A$ induced from $e_1, \cdots, e_n$. By the surjection of $g : A" \to A$ and by the $A"$-flatness of $W_{A",e}$, we can take a basis $\tilde{e}_1, \cdots, \tilde{e}_n$ of $W_{A",e}$ such that the basis

$\bar{\bar{e}}_1, \cdots, \bar{\bar{e}}_n$ of $W_{A",e} \otimes_{A"} A$ induced from $\tilde{e}_1, \cdots, \tilde{e}_n$ satisfies $h(\overline{e}_i) = \bar{\bar{e}}_i$ for any i. If we define by

$$W_e^{'"} := W_{A',e} \times_{W_{A,e}} W_{A",e} := \{(x,y) \in W_{A',e} \times W_{A",e} | h(\bar{x}) = \bar{y}\},$$

then $W_e^{'"}$ is a free $B_e \otimes_{\mathbb{Q}_p} (A' \times_A A")$-module with a basis $(e_1, \tilde{e}_1), \cdots, (e_n, \tilde{e}_n)$. In the same way, we define by $W_{\mathrm{dR}}^{'"+} := W_{A',\mathrm{dR}}^+ \times_{W_{A,\mathrm{dR}}^+} W_{A",\mathrm{dR}}^+$, which is a free $B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} (A' \times_A A")$-module. If we put $W_{A'} \times_{W_A} W_{A"} := (W_e^{'"}, W_{\mathrm{dR}}^{'"+})$, then this is a $(A' \times_A A")$-B-pair which is a deformation of W over $A' \times_A A"$ such that the equivalent class $[W_{A'} \times_{W_A} W_{A"}] \in D_W(A' \times_A A")$ maps $([W_{A'}], [W_{A"}]) \in D_W(A') \times_{D_W(A)} D_W(A")$. Hence, we have proved (H_1).

Finally, we prove that if $g : A" \to A$ is surjective, then the natural map $D_W(A' \times_A A") \to D_W(A') \times_{D_W(A)} D_W(A")$ is bijective, which proves the condition (H_2) and (H_4), hence we can prove the pro-representability of D_W. Let $W_1^{'"}$, $W_2^{'"}$ be deformations of W over $A' \times_A A"$ such that $[W_1^{'"} \otimes_{A' \times_A A"} A'] = [W_2^{'"} \otimes_{A' \times_A A"} A']$ in $D_W(A')$ and $[W_1^{'"} \otimes_{A' \times_A A"} A"] = [W_2^{'"} \otimes_{A' \times_A A"} A"]$ in $D_W(A")$. Under this situation, we want to show $[W_1^{'"}] = [W_2^{'"}]$ in $D_W(A' \times_A A")$. We put $W_{1A'} := W_1^{'"} \otimes_{A' \times_A A"} A'$, $W_{1A"} := W_1^{'"} \otimes_{A' \times_A A"} A"$, $W_{1A} := W_1^{'"} \otimes_{A' \times_A A"} A$, and same for $W_{2A'}$, $W_{2A"}$, W_{2A}, then we have natural isomorphisms $W_1^{'"} \xrightarrow{\sim} W_{1A'} \times_{W_{1A}} W_{1A"}$ and $W_2^{'"} \xrightarrow{\sim} W_{2A'} \times_{W_{2A}} W_{2A"}$ defined as in the last paragraph. Because $[W_{1A'}] = [W_{2A'}]$ and $[W_{1A"}] = [W_{2A"}]$, we have isomorphisms $h' : W_{1A'} \xrightarrow{\sim} W_{2A'}$ and $h" : W_{1A"} \xrightarrow{\sim} W_{2A"}$. By reduction of these isomorphisms, we obtain an automorphism $\bar{h}' \circ \bar{h}^{"-1} : W_{2A} \xrightarrow{\sim} W_{1A} \xrightarrow{\sim} W_{2A}$. By Lemma 2.28 and by the surjection $g : A^{"\times} \to A^\times$, we can find an automorphism $\tilde{h} : W_{2A"} \xrightarrow{\sim} W_{2A"}$ such that $\bar{\tilde{h}} = \bar{h}' \circ \bar{h}^{"-1}$. If we define a morphism

$$h^{'"} : W_{1A'} \times_{W_{1A}} W_{1A"} \to W_{2A"} \times_{W_{2A}} W_{2A'} : (x,y) \mapsto (h_1(x), \tilde{h} \circ h_2(y)),$$

then we can see that this is well-defined and is isomorphism. Hence we finish to prove this proposition. □

Proposition 2.29. *Let $W := (W_e, W_{\mathrm{dR}}^+)$ be an E-B-pair of rank n. If $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$, then the functor D_W is formally smooth.*

Proof. Let $A' \to A$ be a small extension in $\mathcal{C}_E$, we denote the kernel by $I \subseteq A'$. Let $W_A := (W_{e,A}, W_{\mathrm{dR},A}^+)$ be a deformation of W over A, then it suffices to show that there exists an A'-B-pair $W_{A'}$ such that $W_{A'} \otimes_{A'} A \xrightarrow{\sim} W_A$. We fix a basis of $W_{e,A}$ as a $B_e \otimes_{\mathbb{Q}_p} A$-module. Using this basis and the G_K-action on $W_{e,A}$, we obtain a continuous one cocycle

$$\rho_e : G_K \to \mathrm{GL}_n(B_e \otimes_{\mathbb{Q}_p} A).$$

In the same way, if we fix a basis of $W_{\mathrm{dR},A}^+$ as a $B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} A$-module, we obtain a continuous cocycle

$$\rho_{\mathrm{dR}} : G_K \to \mathrm{GL}_n(B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} A).$$

From the canonical isomorphism $W_{e,A} \otimes_{B_e} B_{\mathrm{dR}} \xrightarrow{\sim} W^+_{\mathrm{dR},A} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$, we obtain a matrix $P \in \mathrm{GL}_n(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A)$ such that

$$P\rho_e(g)g(P)^{-1} = \rho_{\mathrm{dR}}(g) \text{ for any } g \in G_K.$$

We fix an E-linear section $s : A \to A'$ of $A' \to A$ and fix a lifting $\tilde{P} \in \mathrm{GL}_n(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A')$ of P. Using this section, we obtain continuous liftings

$$\tilde{\rho}_e := s \circ \rho_e : G_K \to \mathrm{GL}_n(B_e \otimes_{\mathbb{Q}_p} A')$$

of ρ_e and

$$\tilde{\rho}_{\mathrm{dR}} := s \circ \rho_{\mathrm{dR}} : G_K \to \mathrm{GL}_n(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A')$$

of ρ_{dR}. Using these liftings, we define

$$c_e : G_K \times G_K \to I \otimes_E \mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e)$$

by

$$\begin{aligned}\tilde{\rho}_e(g_1g_2)g_1(\tilde{\rho}_e(g_2))^{-1}\tilde{\rho}_e(g_1)^{-1} = 1 + c_e(g_1, g_2) \quad &\in 1 + I \otimes_{A'} \mathrm{M}_n(B_e \otimes_{\mathbb{Q}_p} A') \\ &= 1 + I \otimes_E \mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e)\end{aligned}$$

for any $g_1, g_2 \in G_K$. In the same way, we define

$$c_{\mathrm{dR}} : G_K \times G_K \to I \otimes_E \mathrm{Hom}_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W^+_{\mathrm{dR}}, W^+_{\mathrm{dR}})$$

by

$$\tilde{\rho}_{\mathrm{dR}}(g_1g_2)g_1(\tilde{\rho}_{\mathrm{dR}}(g_2))^{-1}\tilde{\rho}_{\mathrm{dR}}(g_1)^{-1} = 1 + c_{\mathrm{dR}}(g_1, g_2).$$

We define

$$c : G_K \to I \otimes_E \mathrm{Hom}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$$

by

$$\tilde{P}\tilde{\rho}_e(g)g(\tilde{P})^{-1}\tilde{\rho}_{\mathrm{dR}}(g)^{-1} = 1 + c(g) \in 1 + I \otimes_E \mathrm{Hom}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}}).$$

These c_e and c_{dR} are continuous two cocycles, i.e. satisfy

$$g_1c_*(g_2, g_3) - c_*(g_1g_2, g_3) + c_*(g_1, g_2g_3) - c_*(g_1, g_2) = 0$$

for any $g_1, g_2, g_3 \in G_K$ ($* = e, \mathrm{dR}$). Moreover, we can check that c_e and c_{dR} and c satisfy

$$c_e(g_1, g_2) - c_{\mathrm{dR}}(g_1, g_2) = g_1(c(g_2)) - c(g_1g_2) + c(g_1)$$

for any $g_1, g_2, g_3 \in G_K$, here we note that the isomorphism $\mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e) \otimes_{B_e} B_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{Hom}_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W^+_{\mathrm{dR}}, W^+_{\mathrm{dR}}) \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$ is given by $c \mapsto \bar{P}^{-1}c\bar{P}$, where $\bar{P} \in \mathrm{GL}_n(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)$ is the reduction of $P \in \mathrm{GL}_n(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A)$. By the definition of Galois cohomology of B-pairs, these mean that $(c_e, c_{\mathrm{dR}}, c)$ defines an element $[(c_e, c_{\mathrm{dR}}, c)]$ in $I \otimes_E \mathrm{H}^2(G_K, \mathrm{ad}(W))$. Until here, we don't use the condition $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ and we can show that $[(c_e, c_{\mathrm{dR}}, c)]$ doesn't depend on the choice of s or $\tilde{P}$, i.e. depends only on W_A. Under the assumption that $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$, there exists a triple $(f_e, f_{\mathrm{dR}}, f)$ such that $f_e : G_K \to I \otimes_E \mathrm{Hom}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e)$

and $f_{\mathrm{dR}} : G_K \to I \otimes_E \mathrm{Hom}_{B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}(W^+_{\mathrm{dR}}, W^+_{\mathrm{dR}})$ are continuous maps and $f \in I \otimes_E \mathrm{Hom}_{B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$ and these satisfy

$$c_e(g_1, g_2) = g_1 f_e(g_2) - f_e(g_1 g_2) + f_e(g_1)$$

and

$$c_{\mathrm{dR}}(g_1, g_2) = g_1 f_{\mathrm{dR}}(g_2) - f_{\mathrm{dR}}(g_1 g_2) + f_{\mathrm{dR}}(g_1)$$

and

$$c(g_1) = f_{\mathrm{dR}}(g_1) - \bar{P}^{-1} f_e(g_1)\bar{P} + (g_1 f - f)$$

for any $g_1, g_2 \in G_K$. Using these, we define new liftings $\rho'_e : G_K \to \mathrm{GL}_n(B_e \otimes_{\mathbb{Q}_p} A')$ by

$$\rho'_e(g) := (1 + f_e(g))\tilde{\rho}_e(g),$$

$\rho'_{\mathrm{dR}}(g) : G_K \to \mathrm{GL}_n(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A')$ by

$$\rho'_{\mathrm{dR}}(g) := (1 + f_{\mathrm{dR}}(g))\tilde{\rho}_{\mathrm{dR}}(g)$$

and define

$$P' := (1 + f)\tilde{P} \in \mathrm{GL}_n(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A'),$$

then we can check that these satisfy

$$\rho'_e(g_1 g_2) = \rho'_e(g_1) g_1(\rho'_e(g_2)) \text{ and } \rho'_{\mathrm{dR}}(g_1 g_2) = \rho'_{\mathrm{dR}}(g_1) g_1(\rho'_{\mathrm{dR}}(g_2))$$

and

$$P' \rho'_e(g_1) g (P')^{-1} = \rho'_{\mathrm{dR}}(g_1)$$

for any $g_1, g_2 \in G_K$. By the definition of A'-B-pair, this means that $(\rho'_e, \rho'_{\mathrm{dR}}, P')$ defines an A'-B-pair and this is a lift of W_A by the definition. We finish the proof of this proposition. □

Corollary 2.30. *Let W be an E-B-pair of rank n. If $\mathrm{End}_{G_K}(W) = E$ and $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$, then the functor D_W is pro-representable by R_W such that*

$$R_W \xrightarrow{\sim} E[[T_1, \cdots, T_d]] \quad \textit{where } d := [K : \mathbb{Q}_p]n^2 + 1.$$

Proof. The existence and formally smoothness of R_W follows from Proposition 2.26 and Proposition 2.29. For the dimension, we have

$$\begin{aligned} \dim_E D_W(E[\varepsilon]) &= \dim_E \mathrm{H}^1(G_K, \mathrm{ad}(W)) \\ &= [K : \mathbb{Q}_p]n^2 + \dim_E \mathrm{H}^0(G_K, \mathrm{ad}(W)) + \dim_E \mathrm{H}^2(G_K, \mathrm{ad}(W)) \\ &= [K : \mathbb{Q}_p]n^2 + 1 \end{aligned}$$

by Theorem 2.8 and Lemma 2.27. □

2.3. **Trianguline deformations of trianguline B-pairs.** In this subsection, we define the trianguline deformation functor for split trianguline E-B-pairs and prove the pro-representability and the formally smoothness under some conditions and calculate the dimension of the universal deformation ring of this functor. These are the generalizations of Bellaïche-Chenevier's works in the $\mathbb{Q}_p$ case. In § 2 of [Bel-Ch09], Bellaïche-Chenevier proved all of these in the case of $K = \mathbb{Q}_p$ by using (φ, Γ)-modules over the Robba ring and by using Colmez's theory of trianguline representations for $K = \mathbb{Q}_p$ ([Co08]). We generalize their theory by using B-pairs and the theory of trianguline representations for any p-adic field ([Na09] or § 2.1).

We first define the notion of split trianguline A-B-pairs as follows.

Definition 2.31. Let W be a rank n A-B-pairs. We say that W is a split trianguline A-B-pair if there exists a sequence of sub A-B-pairs $\mathcal{T} : 0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ such that W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is a rank one A-B-pair for any $0 \leqq i \leqq n-1$.

By Proposition 2.15, there exist continuous homomorphisms $\delta_i : K^\times \to A^\times$ such that $W_i/W_{i-1} \xrightarrow{\sim} W(\delta_i)$ for any $1 \leqq i \leqq n$. We say that $\{\delta_i\}_{i=1}^n$ is the parameter of the triangulation $\mathcal{T}$.

Next, we define the trianguline deformation functor. Let W be a rank n split trianguline E-B-pair. We fix a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ of W. Under this situation, we define the trianguline deformation as follows.

Definition 2.32. Let A be an object in $\mathcal{C}_E$. We say that $(W_A, \iota, \mathcal{T}_A)$ is a trianguline deformation of $(W, \mathcal{T})$ over A if (W_A, ι) is a deformation of W over A and W_A is a split trianguline A-B-pair with a triangulation $\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n,A} = W_A$ such that $\iota(W_{i,A} \otimes_A E) = W_i$ for any $1 \leqq i \leqq n$. Let $(W_A, \iota, \mathcal{T}_A)$ and $(W'_A, \iota', \mathcal{T}'_A)$ be two trianguline deformations of $(W, \mathcal{T})$ over A. We say that $(W_A, \iota, \mathcal{T}_A)$ and $(W'_A, \iota', \mathcal{T}'_A)$ are equivalent if there exists an isomorphism of A-B-pairs $f : W_A \xrightarrow{\sim} W'_A$ satisfying $\iota = \iota' \circ \bar{f}$ and $f(W_{i,A}) = W'_{i,A}$ for any $1 \leqq i \leqq n$.

Definition 2.33. Let W be a split trianguline E-B-pair with a triangulation $\mathcal{T}$. We define the trianguline deformation functor $D_{W,\mathcal{T}}$ from the category $\mathcal{C}_E$ to the category of sets by

$$D_{W,\mathcal{T}}(A) := \{ \begin{array}{l} \text{equivalent classes } (W_A, \iota, \mathcal{T}_A) \text{ of} \\ \text{trianguline deformations of } (W, \mathcal{T}) \text{ over } A\}. \end{array}$$

for any $A \in \mathcal{C}_E$.

By definition, we have a natural map of functors from $D_{W,\mathcal{T}}$ to D_W by forgetting the triangulation, i.e. defined by

$$D_{W,\mathcal{T}}(A) \to D_W(A) : [(W_A, \iota, \mathcal{T}_A)] \mapsto [(W_A, \iota)].$$

In general, $D_{W,\mathcal{T}}$ is not a sub functor of D_W by this map, i.e. a deformation W_A can have many liftings of the triangulation $\mathcal{T}$. Here, we give a sufficient condition

for $D_{W,\mathcal{T}}$ to be a sub functor of D_W. Let $\{\delta_i\}_{i=1}^n$ be the parameter of triangulation $\mathcal{T}$.

Lemma 2.34. *If we have $\delta_j/\delta_i \neq \prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and for any $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\leqq 0}$, then the functor $D_{W,\mathcal{T}}$ is a sub functor of D_W.*

Proof. Let W_A be a deformation of W over A, let $0 \subseteq W_{A,1} \subseteq \cdots \subseteq W_{A,n-1} \subseteq W_A$ and $0 \subseteq W'_{A,1} \subseteq \cdots \subseteq W'_{A,n-1} \subseteq W_A$ be two triangulations which are lifts of $\mathcal{T}$. It suffices to show that $W_{A,i} = W'_{A,i}$ for any i. By induction, it suffices to show $W_{A,1} = W'_{A,1}$. For proving this, first we consider $\mathrm{Hom}_{G_K}(W_{1,A}, W_A)$. This is equal to $\mathrm{H}^0(G_K, W_{1,A}^\vee \otimes W_A)$. Because $\mathrm{H}^0(G_K, -)$ is left exact and $\mathrm{H}^0(G_K, W(\delta)) = 0$ for any $\delta : K^\times \to E^\times$ such that $\delta \neq \prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\leqq 0}$ by Proposition 2.9, we have

$$\mathrm{H}^0(G_K, W_{1,A}^\vee \otimes (W_{i+1,A}/W_{i,A})) = \mathrm{H}^0(G_K, W_{1,A}^\vee \otimes (W'_{i+1,A}/W'_{i,A})) = 0$$

for any $i \geqq 1$. Hence, we obtain equalities

$$\mathrm{Hom}_{G_K}(W_{1,A}, W_{1,A}) = \mathrm{Hom}_{G_K}(W_{1,A}, W_A) = \mathrm{Hom}_{G_K}(W_{1,A}, W'_{1,A}).$$

This means that the given inclusion $W_{1,A} \hookrightarrow W_A$ factors through $W'_{1,A} \hookrightarrow W_A$. By the same reason, the inclusion $W'_{1,A} \hookrightarrow W_A$ also factors through $W_{1,A} \hookrightarrow W_A$. Hence we obtain an equality $W_{1,A} = W'_{1,A'}$. □

Next, we prove relative representability of $D_{W,\mathcal{T}}$. But, before doing this, we need to define the following functor which is a B-pair version of Lemma 2.3.8 of [Bel-Ch09]. Let $W = (W_e, W_{\mathrm{dR}}^+)$ be an E-B-pair. Then we define by

$F(W) := \{x \in W_e \cap W_{\mathrm{dR}}^+ | \exists n \in \mathbb{Z}_{\geq 1}, (\sigma_1 - 1)(\sigma_2 - 1)\cdots(\sigma_n - 1)x = 0, \forall\sigma_1, \cdots, \sigma_n \in G_K\}$

which is E-vector space with G_K-action, hence F is a left exact functor form the category of E-B-pairs to that of $E[G_K]$-modules. By this definition, we obtain the following lemma, which is the B-pair version of Lemma 2.3.8 of [Bel-Ch09].

Lemma 2.35. *Let $\delta : K^\times \to E^\times$ be a continuous homomorphism, then $F(W(\delta)) \neq 0$ if and only if $\mathrm{H}^0(G_K, W(\delta))$.*

Using this lemma, we prove relative representability of $D_{W,\mathcal{T}}$.

Proposition 2.36. *Let W be a trianguline representations with a triangulation $\mathcal{T}$ such that the parameter $\{\delta_i\}_{i=1}^n$ satisfies $\delta_j/\delta_i \neq \prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and for any $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\leqq 0}$, then the natural map of functors $D_{W,\mathcal{T}} \to D_W$ is relatively representable.*

Proof. By §23 of [Ma97], it suffices to check that the map $D_{W,\mathcal{T}} \to D_W$ satisfies the fallowing three conditions (1), (2), (3).

(1): For any map $A \to A'$ in $\mathcal{C}_E$ and $W_A \in D_{W,\mathcal{T}}(A)$, then $W_A \otimes_A A' \in D_{W,\mathcal{T}}(A')$.

(2): For any maps $A' \to A$ and $A" \to A$ in $\mathcal{C}_E$ and $W'" \in D_W(A' \times_A A")$, if $W'" \otimes_{A'\times_A A"} A' \in D_{W,\mathcal{T}}(A')$ and $W'" \otimes_{A'\times_A A"} A" \in D_{W,\mathcal{T}}(A")$, then $W'" \in D_{W,\mathcal{T}}(A' \times_A A")$.

(3): For any inclusion $A \hookrightarrow A'$ in $\mathcal{C}_E$ and $W_A \in D_W(A)$, if $W_A \otimes_A A' \in D_{W,\mathcal{T}}(A')$, then $W_A \in D_{W,\mathcal{T}}(A)$.

The condition (1) is trivial. For (2), let $W^{'''} \in D_W(A' \times_A A")$ be such that $W_{A'} := W^{'''} \otimes_{A' \times_A A"} A' \in D_{W,\mathcal{T}}(A')$ and $W_{A"} := W^{'''} \otimes_{A' \times_A A"} A" \in D_{W,\mathcal{T}}(A")$. We put $W_A := W^{'''} \otimes_{A' \times_A A"} A$. In the same way as in the proof of Proposition 2.26, we have an isomorphism $W^{'''} \xrightarrow{\sim} W_{A'} \times_{W_A} W_{A"}$. By Lemma 2.34, the triangulations of W_A induced from $W_{A'}$ and $W_{A"}$ are same, hence these triangulations induce a triangulation of $W^{'''} \xrightarrow{\sim} W_{A'} \times_{W_A} W_{A"}$, i.e. $W^{'''} \in D_{W,\mathcal{T}}(A' \times_A A")$.

Finally, we prove the condition (3). The proof is essentially same as that of Proposition 2.3.9 of [Bel-Ch09], but here we give the proof for convenience of readers. Let $W \in D_W(A)$ and $A \hookrightarrow A'$ be an inclusion such that $W_A \otimes_A A' \in D_{W,\mathcal{T}}(A')$. Let $0 \subseteq W_{1,A'} \subseteq \cdots \subseteq W_{n-1,A'} \subseteq W_A \otimes_A A'$ be a triangulation lifting of $\mathcal{T}$. By the induction on the rank of W, it suffices to show that there exists a rank one sub A-B-pair $W_{1,A} \subseteq W_A$ such that $W_{1,A} \otimes_A A' = W_{1,A'}$ and that $W_A/W_{1,A}$ is an A-B-pair. By Proposition 2.15, there exists a continuous homomorphism $\delta_{1,A'} : K^{\times} \to A'^{\times}$ such that $W_{1,A'} \xrightarrow{\sim} W(\delta_{1,A'})$. Twisting W by δ_1^{-1}, we may assume that $\delta_{1,A'} \equiv 1(\mod \mathfrak{m}_{A'})$. Under this assumption, we apply the functor $F(-)$. Because $\delta_{1,A'}$ is unitary, there exists a continuous character $\tilde{\delta}_{1,A'} : G_K^{\mathrm{ab}} \to A'^{\times}$ such that $W(\delta_{1,A'}) \xrightarrow{\sim} W(A'(\tilde{\delta}_{1,A'})) = (B_e \otimes_{\mathbb{Q}_p} A'(\tilde{\delta}_{1,A'}), B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A'(\tilde{\delta}_{1,A'}))$, hence $W_{1,A',e} \cap W^+_{1,A',\mathrm{dR}} = A'(\tilde{\delta}_{1,A'})$. Moreover, because the image of $\delta_{1,A'}$ is in $1 + \mathfrak{m}_{A'}$, we also have $F(W_{1,A'}) = A'(\tilde{\delta}_{1,A'})$. Next, because $(W_A \otimes_A A')/W_{1,A'}$ is a successive extension of $W(\delta_i\delta_1^{-1})$ $(i \geqq 2)$ as an E-B-pair, the left exactness of F implies that $F((W_A \otimes_A A')/W_{1,A'}) = 0$ by Lemma 2.35. Applying F to the short exact sequence $0 \to W_{1,A'} \to W_A \otimes_A A' \to (W_A \otimes_A A')/W_{1,A'} \to 0$, we obtain $A'(\tilde{\delta}_{1,A'}) \xrightarrow{\sim} F(W_{1,A'}) = F(W_A \otimes_A A')$. In the same way, we obtain $E = F(W_1) = F(W)$. Then, by left exactness and by considering the length, we can show that $F(W_A)$ is a free A-module of rank one and that the natural map $F(W_A) \to F(W)$ (induced by the natural quotient map $W_A \to W$) is surjection and that the natural map $F(W_A) \otimes_A A' \to F(W_A \otimes_A A')$ is isomorphism. If we define $\tilde{\delta}_{1,A} : G_K^{\mathrm{ab}} \to A^{\times}$ such that $F(W_A) \xrightarrow{\sim} A(\tilde{\delta}_{1,A})$ and define $W_{1,A}$ as the image of the natural map $(B_e \otimes_{\mathbb{Q}_p} F(W_A), B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} F(W_A)) \to W_A$ induced form $F(W) \hookrightarrow W_{A,e}, F(W) \hookrightarrow W^+_{A,\mathrm{dR}}$, then we can check that $W_{1,A}$ is a rank one A-B-pair and that the quotient $W_A/W_{1,A}$ is also an A-B-pair and that $W_{1,A} \otimes_A A' \xrightarrow{\sim} W_{1,A'}$, which proves the condition (3), hence we finish to prove this proposition.

□

Corollary 2.37. *Let W be a trianguline E-B-pair with a triangulation $\mathcal{T}$ such that* $\mathrm{End}_{G_K}(W) = E$ *and the parameter $\{\delta_i\}_{i=1}^n$ of $\mathcal{T}$ satisfies $\delta_j/\delta_i \neq \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\leqq 0}$, then the functor $D_{W,\mathcal{T}}$ is pro-representable by a quotient $R_{W,\mathcal{T}}$ of R_W.*

Proof. This follows from Proposition 2.26 and Proposition 2.36. □

Next, we prove the formally smoothness of the functor $D_{W,\mathcal{T}}$.

Proposition 2.38. *Let W be a trianguline E-B-pair of rank n with a triangulation $\mathcal{T}$ such that whose parameter $\{\delta_i\}_{i=1}^{n}$ satisfies $\delta_i/\delta_j \neq |\mathrm{N}_{K/\mathbb{Q}_p}|\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and for any $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\geqq 1}$, then the functor $D_{W,\mathcal{T}}$ is formally smooth.*

Proof. We prove this proposition by the induction of the rank of W. When W is rank one, then $D_{W,\mathcal{T}} = D_W$ and $\mathrm{ad}(W) = B_E$ is the trivial E-B-pair. Hence $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ by Proposition 2.9. Hence, $D_{W,\mathcal{T}}$ is formally smooth by Proposition 2.29. Let's assume that the proposition is proved for all the rank $n-1$ E-B-pairs, let W be a rank n E-B-pair with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ whose parameter $\{\delta_i\}_{i=1}^{n}$ satisfying the condition as above. Let $A' \to A$ be a small extension in $\mathcal{C}_E$ and let W_A be a trianguline deformation of $(W, \mathcal{T})$ with a triangulation $\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n-1,A} \subseteq W_{n,A} = W_A$ which is a lift of $\mathcal{T}$. It suffices to prove that there exists a split trianguline A'-B-pair $W_{A'}$ with a triangulation $0 \subseteq W_{1,A'} \subseteq \cdots \subseteq W_{n-1.A'} \subseteq W_{n,A'} = W_{A'}$ which is a lift of W_A and $\mathcal{T}_A$. We take a lifting as follows. First, because W_{n-1} is a trianguline E-B-pair of rank $n-1$ satisfying the condition as above, hence there exists a rank $n-1$ trianguline A'-B-pair $W_{n-1,A'}$ with a triangulation $0 \subseteq W_{1,A'} \subseteq \cdots \subseteq W_{n-2,A'} \subseteq W_{n-1,A'}$ which is a lift of $W_{n-1,A}$ and $0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n-1,A}$ by the induction hypothesis. If we put $\mathrm{gr}_n W_A := W_A/W_{n-1,A}$, then by the rank one case and by Proposition 2.15, there exists a continuous homomorphism $\delta_{n,A'} : K^\times \to A'^\times$ such that the rank one A'-B-pair $W(\delta_{n,A'})$ satisfies $W(\delta_{n,A'}) \otimes_{A'} A = W(\delta_{n,A}) \xrightarrow{\sim} \mathrm{gr}_n W_A$, where $\delta_{n,A} : K^\times \to A^\times$ is the reduction of $\delta_{n,A'}$. We can see the isomorphism class $[W_A]$ as an element in $\mathrm{Ext}^1(W(\delta_{n,A}), W_{n-1,A}) \xrightarrow{\sim} \mathrm{H}^1(G_K, W_{n-1,A}(\delta_{n,A}^{-1}))$. If we take the long exact sequence associated to

$$0 \to I \otimes_E W_{n-1}(\delta_n^{-1}) \to W_{n-1,A'}(\delta_{n,A'}^{-1}) \to W_{n-1,A}(\delta_{n,A}^{-1}) \to 0$$

where $I \subseteq A'$ is the kernel of $A' \to A$, then we obtain a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{H}^1(G_K, W_{n-1,A'}(\delta_{n,A'}^{-1})) &\to \mathrm{H}^1(G_K, W_{n-1,A}(\delta_{n,A}^{-1})) \\ &\to I \otimes_E \mathrm{H}^2(G_K, W_{n-1}(\delta_n^{-1})) \to \cdots \end{aligned}$$

By the assumption on $\{\delta_i\}_{i=1}^{n}$ and by Proposition 2.9, we have $\mathrm{H}^2(G_K, W_{n-1}(\delta_n^{-1})) = 0$. Hence we can take a $[W_{A'}] \in \mathrm{Ext}^1(W(\delta_{n,A'}), W_{n-1,A'}) \xrightarrow{\sim} \mathrm{H}^1(G_K, W_{n-1,A'}(\delta_{n,A'}^{-1}))$ which is a trianguline lift of $[W_A]$. This proves the proposition. $\square$

Next, we calculate the dimension of $D_{W,\mathcal{T}}$. For this, we interpret the tangent space $D_{W,\mathcal{T}}(E[\varepsilon])$ in terms of Galois cohomology of B-pair as in Lemma 2.27. Let W be a trianguline E-B-pair with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ whose parameter is $\{\delta_i\}_{i=1}^{n}$, then we define an E-B-pair $\mathrm{ad}_{\mathcal{T}}(W)$ by

$$\mathrm{ad}_{\mathcal{T}}(W) := \{f \in \mathrm{ad}(W) | f(W_i) \subseteq W_i \text{ for any } 1 \leqq i \leqq n\}.$$

Lemma 2.39. *Let W be a trianguline E-B-pair, then there exists a canonical bijection of sets*

$$D_{W,\mathcal{T}}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W)).$$

In particular, if $D_{W,\mathcal{T}}$ has a canonical structure of E-vector space (c.f. the condition (2) in Schlessinger's criterion 2.25), then this bijection is an E-linear isomorphism.

Proof. The construction of the map $D_{W,\mathcal{T}}(E[\varepsilon]) \to \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W))$ is same as in the proof of Lemma 2.27. We put $\mathrm{ad}_{\mathcal{T}}(W) := (\mathrm{ad}_{\mathcal{T}}(W_e), \mathrm{ad}_{\mathcal{T}}(W^+_{\mathrm{dR}}))$. Let $W_{E[\varepsilon]} := (W_{e,E[\varepsilon]}, W^+_{\mathrm{dR},E[\varepsilon]})$ be a trianguline deformation of $(W, \mathcal{T})$ over $E[\varepsilon]$ whose triangulation $\mathcal{T}_{E[\epsilon]}$ is $0 \subseteq W_{1,E[\varepsilon]}, \subseteq \cdots \subseteq W_{n-1,E[\varepsilon]} \subseteq W_{n,E[\varepsilon]} = W_{E[\varepsilon]}$, then we can take a splitting $W_{e,E[\varepsilon]} = W_e e_1 \oplus W_e e_2$ as a filtered $B_e \otimes_{\mathbb{Q}_p} E$-module such that $W_e e_1 = \varepsilon W_{e,E[\varepsilon]}$ and the natural map $W_e e_2 \hookrightarrow W_{e,E[\varepsilon]} \to W_{e,E[\varepsilon]}/\varepsilon W_{e,E[\varepsilon]} \xrightarrow{\sim} W_e$ is such that $ye_2 \mapsto y$ for any $y \in W_e$. If we define $c_e : G_K \to \mathrm{Hom}_{B_e\otimes_{\mathbb{Q}_p}E}(W_e, W_e)$ in the same way as in the proof of Lemma 2.27, we can check that the image of c_e is contained in $\mathrm{ad}_{\mathcal{T}}(W_e)$. In the same way, we can define $c_{\mathrm{dR}} : G_K \to \mathrm{ad}_{\mathcal{T}}(W^+_{\mathrm{dR}})$ from a filtered splitting $W^+_{\mathrm{dR},E[\varepsilon]} = W^+_{\mathrm{dR}} e_1 \oplus W^+_{\mathrm{dR}} e'_2$. Moreover, we can define $c \in \mathrm{ad}_{\mathcal{T}}(W_{\mathrm{dR}})$ by $ye_2 = c(y)e_1 + ye'_2$ for any $y \in W_{\mathrm{dR}}$. Then the map

$$D_{W,\mathcal{T}}(E[\varepsilon]) \to \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W)) : [(W_{E[\varepsilon]}, \mathcal{T}_{E[\varepsilon]})] \mapsto [(c_e, c_{\mathrm{dR}}, c)]$$

defines a bijection and, when $D_{W,\mathcal{T}}(E[\varepsilon])$ has a canonical E-vector space structure, this is an E-linear isomorphism. □

We calculate the dimension of $R_{W,\mathcal{T}}$.

Proposition 2.40. *Let W be a split trianguline E-B-pair of rank n with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$. We assume that $(W, \mathcal{T})$ satisfies the following conditions,*

(0) $\mathrm{End}_{G_K}(W) = E$,
(1) *For any* $1 \leqq i < j \leqq n$, $\delta_j/\delta_i \neq \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *for any* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$,
(2) *For any* $1 \leqq i < j \leqq n$, $\delta_i/\delta_j \neq |\mathrm{N}_{K/\mathbb{Q}_p}| \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *for any* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 1}$,

then the universal trianguline deformation ring $R_{W,\mathcal{T}}$ is a quotient ring of R_W such that

$$R_{W,\mathcal{T}} \xrightarrow{\sim} E[[T_1, \cdots, T_{d_n}]] \text{ where } d_n := \frac{n(n+1)}{2}[K : \mathbb{Q}_p] + 1.$$

Proof. From Proposition 2.36 and Proposition 2.38 and Lemma 2.39, it suffices to show that $\dim_E \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = d_n$. We prove this by the induction on the rank n of W. When $n = 1$, then $\mathrm{ad}_{\mathcal{T}}(W) = \mathrm{ad}(W) = B_E$, hence the proposition follows from Proposition 2.9. Let W be a rank n trianguline E-B-pair as above, let $\mathcal{T}_{n-1} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-2} \subseteq W_{n-1}$ be the triangulation of $W_{n-1} (\subseteq W)$. Then

for any $f \in \mathrm{ad}_{\mathcal{T}}(W)$, the restriction of f to W_{n-1} is an element of $\mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1})$ and this defines a short exact sequence of E-B-pair

$$0 \to \mathrm{Hom}(W(\delta_n), W) \to \mathrm{ad}_{\mathcal{T}}(W) \to \mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1}) \to 0.$$

From this, we have

$$\begin{aligned} \mathrm{rank}(\mathrm{ad}_{\mathcal{T}}(W)) &= \mathrm{rank}(\mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1})) + n \\ &= 1 + 2 + \cdots + n = \tfrac{n(n+1)}{2}. \end{aligned}$$

From this and Theorem 2.8, it suffices to show that $\mathrm{H}^0(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = E$ and $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = 0$. For H^0, this follows from

$$E \subseteq \mathrm{H}^0(G_K, \mathrm{ad}_{\mathcal{T}}(W)) \subseteq \mathrm{H}^0(G_K, \mathrm{ad}(W)) = E.$$

We prove $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = 0$ by the induction of the rank of W. When $n = 1$, this follows from Proposition 2.9. When W is rank n, then from the above short exact sequence, we have the following long exact sequence

$$\cdots \to \mathrm{H}^2(G_K, \mathrm{Hom}(W(\delta_n), W)) \to \mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) \to \mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1})) \to 0,$$

then we have $\mathrm{H}^2(G_K, \mathrm{Hom}(W(\delta_n), W)) = 0$ by Proposition 2.9 and by the assumption on $\{\delta_i\}_{i=1}^n$. Hence, by the induction hypothesis, we have $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = 0$. We finish the proof of this proposition.

□

2.4. **Deformations of benign B-pairs.** In this final subsection, we study benign representations (more generally benign B-pairs) which is a class of potentially crystalline and trianguline representations and have some very good properties for trianguline deformations and play a crucial role in Zariski density problem of modular Galois (or crystalline) representations in some deformation spaces of global (or local) p-adic Galois representations. This class was defined by Kisin in the case of $K = \mathbb{Q}_p$ and the rank W is 2 in [Ki03] and [Ki10]. He studied some deformation theoretic properties of this class in [Ki03] and used these in a crucial way in his proof of Zariski density of two dimensional crystalline representations of $G_{\mathbb{Q}_p}$. For higher dimensional and the $\mathbb{Q}_p$ case, Bellaïche-Chenevier ([Bel-Ch09]) and Chenevier ([Ch09b]) were the first ones who noticed the importance of benign representations in the study of p-adic families of trianguline representations. In particular, Chenevier ([Ch09b], where he calls "generic" instead of benign) discovered and proved a crucial theorem concerning to tangent spaces of the universal deformation rings of benign representations. For the two dimensional case, this was discovered by Kisin ([Ki10]) implicitly when $K = \mathbb{Q}_p$. In fact, by using this theorem, Chenevier ([Ch09b]) proved many kinds of theorems concerning Zariski density of modular Galois representations in some deformation spaces of global p-adic representations.

The aim of this subsection is to generalize the definition of benign representations and the Chenevier's theorem for any K-case.

2.4.1. *Benign B-pairs.* Let $P(X) \in \mathcal{O}_K[X]$ be a polynomial such that $P(X) \equiv \pi_K X$ (mod deg 2) and that $P(X) \equiv X^q$ (mod π_K), where $q := p^f$ and $f := [K_0 : \mathbb{Q}_p]$. We take the Lubin-Tate's formal group law $\mathcal{F}$ of K such that $[\pi_K] = P(X)$, where $[-] : \mathcal{O}_K \xrightarrow{\sim} \mathrm{End}(\mathcal{F})$. We denote by K_n the abelian extension of K generated by $[\pi_K^n]$-torsion points of $\mathcal{F}$ for any n, then we have a canonical isomorphism $\chi_{\mathrm{LT},n} : \mathrm{Gal}(K_n/K) \xrightarrow{\sim} (\mathcal{O}_K^{\times}/\pi^n\mathcal{O}_K)^{\times}$. We put $K_{\mathrm{LT}} := \cup_{n=1}^{\infty} K_n$ and $G_n := \mathrm{Gal}(K_n/K)$.

In [Ki10], [Bel-Ch09] or [Ch09b] etc, benign representation is defined as a special class of crystalline representations. But, as we show in the sequel, we can easily generalize the main theorem to some potentially crystalline representations. Hence, before defining benign representations, we first define the following class of potentially crystalline representations.

Definition 2.41. Let W be an E-Bpair. We say that W is crystabelline if $W|_{G_L}$ is a crystalline E-B-pair of G_L for a finite abel extension L of K.

Remark 2.42. Because a finite abel extension L of K is contained in $K_m L'$ for some $m \geqq 0$, where L' is a finite unramified extension of K, by using Hilbert 90, we can easily show that W is crystabelline if and only if $W|_{G_{K_m}}$ is crystalline for some $m \geqq 0$.

Let W be a crystabelline E-B-pair of rank n such that $W|_{G_{K_m}}$ is crystalline for some m. Let $D^{K_m}_{\mathrm{cris}}(W) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} W)^{G_{K_m}}$ be the rank n E-filtered (φ, G_m)-module over K. Because K_m is totally ramified over K, this is a free $K_0 \otimes_{\mathbb{Q}_p} E$-module of rank n. We take an embedding $\sigma : K_0 \hookrightarrow \overline{E}$. This defines a map $\sigma : K_0 \otimes_{\mathbb{Q}_p} E \to \overline{E} : x \otimes y \to \sigma(x)y$. From this, we can define the σ-component

$$D^{K_m}_{\mathrm{cris}}(W)_\sigma := D^{K_m}_{\mathrm{cris}}(W) \otimes_{K_0 \otimes_{\mathbb{Q}_p} E, \sigma} \overline{E},$$

this has a $\overline{E}$-linear φ^f-action and a $\overline{E}$-linear G_m-action. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a solution in $\overline{E}$ (with multiplicities) of $\det_{\overline{E}}(T \cdot \mathrm{id} - \varphi^f|_{D^{K_m}_{\mathrm{cris}}(W)_\sigma}) \in \overline{E}[T]$. Because φ^f and the G_m-action commute, any generalized φ^f-eigenvector subspaces of $D^{K_m}_{\mathrm{cris}}(W)_\sigma$ are preserved by the action of G_m. Hence we can take a $\overline{E}$-basis $e_{1,\sigma}, \cdots, e_{n,\sigma}$ of $D^{K_m}_{\mathrm{cris}}(W)_\sigma$ such that $e_{i,\sigma}$ is a generalized eigenvector of φ^f with an eigenvalue $\alpha_i \in \overline{E}^{\times}$ and G_m acts on $e_{i,\sigma}$ by a character $\tilde{\delta}_i : G_m \to \overline{E}^{\times}$ for any i. We change numbering of $\{\alpha_1, \cdots, \alpha_n\}$ so that $e_{1,\sigma}, e_{2,\sigma}, \cdots, e_{n,\sigma}$ gives a φ^f-Jordan decomposition of $D^{K_m}_{\mathrm{cris}}(W)_\sigma$ by this order. Because $\{\sigma, \varphi^{-1}\sigma, \cdots, \varphi^{-(f-1)}\sigma\} = \mathrm{Hom}_{\mathbb{Q}_p}(K_0, \overline{E})$ and

$$\varphi^i : D^{K_m}_{\mathrm{cris}}(W)_\sigma \xrightarrow{\sim} D^{K_m}_{\mathrm{cris}}(W)_{\varphi^{-i}\sigma} : x \otimes y \mapsto \varphi^i(x) \otimes y$$

(for any $x \in D^{K_m}_{\mathrm{cris}}(W)$ and $y \in \overline{E}$) is a $\overline{E}[\varphi^f, G_m]$-isomorphism, the set $\{\alpha_1, \cdots, \alpha_n\}$ doesn't depend on the choice of $\sigma : K_0 \hookrightarrow \overline{E}$. If we put

$$e_i := e_{i,\sigma} + \varphi(e_{i,\sigma}) + \cdots + \varphi^{f-1}(e_{i,\sigma}) \in D^{K_m}_{\mathrm{cris}}(W) \otimes_E \overline{E},$$

we have

$$D^{K_m}_{\mathrm{cris}}(W) \otimes_E \overline{E} = K_0 \otimes_{\mathbb{Q}_p} \overline{E} e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} \overline{E} e_n$$

such that the subspace $K_0 \otimes_{\mathbb{Q}_p} \overline{E}e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} \overline{E}e_i$ is preserved by φ and by the G_m-action for any i. Moreover, if we take a sufficiently large finite extension E' of E, then we can take $e_i \in D^{K_m}_{\mathrm{cris}}(W) \otimes_E E'$ such that

$$D^{K_m}_{\mathrm{cris}}(W) \otimes_E E' = K_0 \otimes_{\mathbb{Q}_p} E'e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} E'e_n$$

and $\alpha_i \in E'$ and $\tilde{\delta}_i : G_m \to E'^{\times}$ for any i.

Using these arguments, we first study a relation between crystabelline E-B-pairs and trianguline E-B-pairs.

Lemma 2.43. *Let W be an E-B-pair of rank n. The following conditions are equivalent,*

(1) *W is crystabelline,*

(2) *W is trianguline (i.e, $W \otimes_E E'$ is a split trianguline E'-B-pair for a finite extension E' of E) and potentially crystalline.*

Proof. First we assume that W is crystabelline. By the above argument, for a sufficiently large finite extension E' of E, we have $D^{K_m}_{\mathrm{cris}}(W) \otimes_E E' = K_0 \otimes_{\mathbb{Q}_p} E'e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} E'e_n$ as above and $K_0 \otimes_{\mathbb{Q}_p} E'e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} E'e_i$ is a sub E'-filtered (φ, G_m)-module of $D^{K_m}_{\mathrm{cris}}(W \otimes_E E')$ for any i. Hence $W \otimes_E E'$ is split trianguline and potentially crystalline by Theorem 2.5.

Next we assume that W is trianguline and potentially crystalline. By extending the coefficient, we may assume that W is split trianguline. We take a triangulation $0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W$ of W. Because sub objects and quotients of crystalline B-pairs are again crystalline, W_i and W_i/W_{i-1} are all potentially crystalline. Because W_i/W_{i-1} are rank one, $W_i/W_{i-1}|_{G_m}$ are crystalline for any i for sufficiently large m. We claim that $W|_{G_m}$ is also crystalline. We prove this claim by the induction on rank n of W. When $n = 1$, this is trivial. We assume that the claim is proved for the rank $n-1$ case, hence $W_{n-1}|_{G_m}$ is crystalline. If we put $W/W_{n-1} \xrightarrow{\sim} W(\delta_n)$, we have $[W] \in \mathrm{H}^1(G_K, W_{n-1}(\delta_n^{-1}))$. By the assumption, there exists a finite Galois extension L of K_m such that $[W]$ is contained in $\mathrm{Ker}(\mathrm{H}^1(G_K, W_{n-1}(\delta_n^{-1})) \to \mathrm{H}^1(G_L, B_{\mathrm{cris}} \otimes_{B_e} (W_{n-1}(\delta_n^{-1}))_e))$. Hence, it suffices to prove that the natural map $\mathrm{H}^1(G_{K_m}, B_{\mathrm{cris}} \otimes_{B_e} (W_{n-1}(\delta_n^{-1}))_e) \to \mathrm{H}^1(G_L, B_{\mathrm{cris}} \otimes_{B_e} (W_{n-1}(\delta_n^{-1}))_e)$ is injective. By inflation restriction sequence, the kernel of this map is $\mathrm{H}^1(\mathrm{Gal}(L/K_m), D^L_{\mathrm{cris}}(W_{n-1}(\delta_n^{-1}))) = 0$. Hence $W|_{G_m}$ is crystalline, i.e. W is crystabelline.

□

From here, we consider a crystabelline E-B-pair W such that $D^{K_m}_{\mathrm{cris}}(W) \xrightarrow{\sim} K_0 \otimes_{\mathbb{Q}_p} Ee_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_n$ such that $K_0 \otimes_{\mathbb{Q}_p} Ee_i$ is preserved by (φ, G_m) and $\varphi^f(e_i) = \alpha_i e_i$ for some $\alpha_i \in E^{\times}$ such that $\alpha_i \neq \alpha_j$ for any $i \neq j$. Let $\mathfrak{S}_n$ be the n-th permutation group. For any $\tau \in \mathfrak{S}_n$, we define a filtration by E-filtered (φ, G_m)-modules on $D^{K_m}_{\mathrm{cris}}(W)$ as follows,

$$\mathcal{F}_\tau : 0 \subseteq F_{\tau,1} \subseteq \cdots \subseteq F_{\tau,n-1} \subseteq F_{\tau,n} = D^{K_m}_{\mathrm{cris}}(W)$$

such that

$$F_{\tau,i} := K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(1)} \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(i)}$$

for any $1 \leqq i \leqq n$, where the filtration on $F_{\tau,i}$ is induced from that on $D^{K_m}_{\mathrm{cris}}(W)$. We put $\mathrm{gr}_{\tau,i}D^{K_m}_{\mathrm{cris}}(W) := F_{\tau,i}/F_{\tau,i-1}$ for any $1 \leqq i \leqq n$. By Theorem 2.5, there exists a filtration

$$\mathcal{T}_\tau : 0 \subseteq W_{\tau,1} \subseteq \cdots \subseteq W_{\tau,n-1} \subseteq W_{\tau,n} = W$$

such that $W_{\tau,i}|_{G_m}$ is crystalline and

$$D^{K_m}_{\mathrm{cris}}(W_{\tau,i}) = F_{\tau,i}.$$

$W_{\tau,i}/W_{\tau,i-1}$ is a rank one crystabelline E-B-pair such that $D^{K_m}_{\mathrm{cris}}(W_{\tau,i}/W_{\tau,i-1}) \xrightarrow{\sim} \mathrm{gr}_{\tau,i}D^{K_m}_{\mathrm{cris}}(W)$. Hence, by Lemma 4.1 of [Na09] and by its proof, there exists $\{k_{(\tau,i),\sigma}\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}$ and homomorphisms $\tilde{\delta}_i : K^\times \to E^\times$ satisfying $\tilde{\delta}_i|_{1+\pi_K^m\mathcal{O}_K} = 1$ and $\tilde{\delta}_i(\pi_K) = 1$, such that $W_{\tau,i}/W_{\tau,i-1} \xrightarrow{\sim} W(\delta_{\alpha_{\tau(i)}}\tilde{\delta}_{\tau(i)}\prod_{\sigma\in\mathcal{P}}\sigma^{k_{(\tau,i),\sigma}})$ for any $1 \leqq i \leqq n$, where $\delta_{\alpha_i} : K^\times \to E^\times$ is the homomorphism such that $\delta_{\alpha_i}|_{\mathcal{O}_K^\times} = 1$ and $\delta_{\alpha_i}(\pi_K) = \alpha_i$. For any $\sigma \in \mathcal{P}$, the set $\{k_{(\tau,1),\sigma}, k_{(\tau,2),\sigma}, \cdots, k_{(\tau,n),\sigma}\}$ is independent of $\tau \in \mathfrak{S}_n$ because these numbers are the σ-part of the Hodge-Tate weights of W. We denote this set (with multiplicities) by $\{k_{1,\sigma}, k_{2,\sigma}, \cdots, k_{n,\sigma}\}$ such that $k_{1,\sigma} \geqq k_{2,\sigma} \geqq \cdots \geqq k_{n,\sigma}$ for any $\sigma \in \mathcal{P}$. Under these notations, we define the notion of benign E-B-pair as follows.

Definition 2.44. Let W be a rank n crystabelline E-B-pair as above. We say that W is a benign E-B-pair if the following conditions hold:

1. For any $i \neq j$, we have $\alpha_i/\alpha_j \neq 1, p^f, p^{-f}$.
2. For any $\sigma \in \mathcal{P}$, we have $k_{1,\sigma} > k_{2,\sigma} > \cdots > k_{n-1,\sigma} > k_{n,\sigma}$.
3. For any $\tau \in \mathfrak{S}_n$ and for any $\sigma \in \mathcal{P}$, we have $k_{(\tau,i),\sigma} = k_{i,\sigma}$ for any $1 \leqq i \leqq n$.

We say that the triangulation $\mathcal{T}_\tau$ is non-critical if $\mathcal{T}_\tau$ satisfies the condition (3).

Remark 2.45. By definition, if W is a benign, then we have $W_{\tau,i}/W_{\tau,i-1} \xrightarrow{\sim} W(\delta_{\alpha_{\tau(i)}}\tilde{\delta}_{\tau(i)}\prod_{\sigma\in\mathcal{P}}\sigma^{k_{i,\sigma}})$ for any $\tau \in \mathfrak{S}_n$ and $1 \leqq i \leqq n$.

Lemma 2.46. *Let W be a benign E-B-pair. If W_1 is a saturated sub E-B-pair of W, then W_1 and W/W_1 are also benign E-B-pairs.*

Proof. This follows from the definition and the fact that sub objects and quotients of crystabelline E-B-pairs are crystabelline E-B-pairs. □

2.4.2. *Deformations of benign B-pairs.*

Lemma 2.47. *Let W be a potentially crystalline E-B-pair satisfying the condition (1) of Definition 2.44, then we have $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ and $(W, \mathcal{T}_\tau)$ satisfies the conditions in Proposition 2.40 except the condition $\mathrm{End}_{G_K}(W) = E$ for any $\tau \in \mathfrak{S}_n$.*

Proof. That $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ follows from the condition (1) of Definition 2.44 and from (2) of Proposition 2.9 because $\mathrm{ad}(W)$ is split trianguline whose graded components are of the forms $(W_{\tau,i}/W_{\tau,i-1}) \otimes (W_{\tau,j}/W_{\tau,j-1})^{\vee}$ for any fixed $\tau \in \mathfrak{S}_n$. Other statements follow from the condition (1) of Definition 2.44. $\square$

Lemma 2.48. *Let W be a benign E-B-pair of rank n, then $\mathrm{End}_{G_K}(W) = E$.*

Proof. We prove this by the induction on n, the rank of W. If $n = 1$, $\mathrm{End}_{G_K}(W) = \mathrm{H}^0(G_K, B_E) = E$. We assume that the lemma is proved for the rank $n-1$ case, let W be a rank n benign E-B-pair. We take an element $\tau \in \mathfrak{S}_n$ and consider the filtration $\mathcal{T}_\tau : 0 \subseteq W_{\tau,1} \subseteq \cdots \subseteq W_{\tau,n-1} \subseteq W_{\tau,n} = W$. By Lemma 2.46, $W_{\tau,n-1}$ is benign of rank $n-1$, hence by the induction hypothesis, we have $\mathrm{End}_{G_K}(W_{\tau,n-1}) = E$. Let $f : W \to W$ be a non-zero morphism of E-B-pairs. By (1) of Definition 2.44 and by Proposition 2.9, we have $\mathrm{Hom}_{G_K}(W_{\tau,n-1}, W/W_{\tau,n-1}) = 0$. Hence we have $f(W_{\tau,n-1}) \subseteq W_{\tau,n-1}$. Because $\mathrm{End}_{G_K}(W_{\tau,n-1}) = E$, we have $f|_{W_{\tau,n-1}} = a \cdot \mathrm{id}_{W_{\tau,n-1}}$ for some $a \in E$. If $a = 0$, then $f : W \to W$ factors through a non-zero morphism $f' : W/W_{\tau,n-1} \to W$. Because $\mathrm{Hom}_{G_K}(W/W_{\tau,n-1}, W_{\tau,n-1}) = 0$ by (1) of Definition 2.44 and by Proposition 2.9, the natural map $\mathrm{Hom}_{G_K}(W/W_{\tau,n-1}, W) \hookrightarrow \mathrm{Hom}_{G_K}(W/W_{\tau,n-1}, W/W_{\tau,n-1}) = E$ is injective, hence the composition of f' with the natural projection $W \to W/W_{\tau,n-1}$ induces an isomorphism $W/W_{\tau,n-1} \xrightarrow{\sim} W/W_{\tau,n-1}$. This implies that the short exact sequence $0 \to W_{\tau,n-1} \to W \to W/W_{\tau,n-1} \to 0$ splits. If we take a section $s : W/W_{\tau,n-1} \hookrightarrow W$, then we can choose $\tau' \in \mathfrak{S}_n$ such that $W_{\tau',1} = s(W/W_{\tau,n-1})$, then this τ' doesn't satisfy the condition (3) in the definition of benign B-pairs. It's contradiction. Hence the above a must not be zero. If $a \neq 0$, consider the map $f - a \cdot \mathrm{id}_W \in \mathrm{End}_{G_K}(W)$, then the same argument as above implies that $f = a \cdot \mathrm{id}_W$. Hence we obtain $\mathrm{End}_{G_K}(W) = E$. $\square$

Corollary 2.49. *Let W be a benign E-B-pair of rank n. The functor D_W is pro-representable by R_W which is formally smooth of dimension $n^2[K : \mathbb{Q}_p] + 1$. For any $\tau \in \mathfrak{S}_n$, the functor $D_{W,\mathcal{T}_\tau}$ is pro-representable by a quotient $R_{W,\mathcal{T}_\tau}$ of R_W which is formally smooth of dimension $\frac{n(n+1)}{2}[K : \mathbb{Q}_p] + 1$.*

Proof. This follows from Proposition 2.40. $\square$

Next, we want to consider the relation between R_W and $R_{W,\mathcal{T}_\tau}$ for all $\tau \in \mathfrak{S}_n$. In particular, we want to compare the tangent space of R_W and the sum of tangent spaces of $R_{W,\mathcal{T}_\tau}$ for all $\tau \in \mathfrak{S}_n$. For this, first we need to recall the potentially crystalline deformation functor.

Definition 2.50. Let W be a potentially crystalline E-B-pair. We define the potentially crystalline deformation functor D_W^{cris} which is a sub functor of D_W defined by

$$D_W^{\mathrm{cris}}(A) := \{[W_A] \in D_W(A) | W_A \text{ is potentially crystalline } \}$$

for any $A \in \mathcal{C}_E$.

Lemma 2.51. *Let W be a potentially crystalline E-B-pair. If $\mathrm{End}_{G_K}(W) = E$, then D_W^{cris} is pro-representable by a quotient R_W^{cris} of R_W which is formally smooth of dimension equal to*

$$\dim_E(D_{\mathrm{dR}}(\mathrm{ad}(W))/\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathrm{ad}(W))) + \dim_E(\mathrm{H}^0(G_K, \mathrm{ad}(W))).$$

Proof. For pro-representability, by Proposition 2.29, it suffices to relatively representability of $D_W^{\mathrm{cris}} \hookrightarrow D_W$ as in the proof of Proposition 2.36. In this case, the conditions (1) and (2) are trivial and (3) follows from the fact that the sub object of potentially crystalline E-B-pair is again potentially crystalline. Formally smoothness follows from Proposition 3.1.2 and Lemma 3.2.1 of [Ki08] by using the deformations of filtered (φ, G_K)-modules. For the dimension, we take a finite Galois extension L of K such that $W|_{G_L}$ is crystalline. By the same argument as in the proof of Lemma 2.43, any $W_A \in D_W^{\mathrm{cris}}(A)$ is crystalline when restricted to G_L. It's easy to check that the map $D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W))$ induces isomorphism $D_W^{\mathrm{cris}}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{H}^1(G_K, \mathrm{ad}(W)) \to \mathrm{H}^1(G_L, B_{\mathrm{cris}} \otimes_{B_e} \mathrm{ad}(W)_e))$. In the same way as in the proof of Lemma 2.43, the natural map $\mathrm{H}^1(G_K, B_{\mathrm{cris}} \otimes_{B_e} \mathrm{ad}(W)_e) \to \mathrm{H}^1(G_L, B_{\mathrm{cris}} \otimes_{B_e} \mathrm{ad}(W)_e)$ is injective. Hence we have an isomorphism

$$D_W^{\mathrm{cris}}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{H}^1(G_K, \mathrm{ad}(W)) \to \mathrm{H}^1(G_K, B_{\mathrm{cris}} \otimes_{B_e} \mathrm{ad}(W)_e)).$$

We can calculate the dimension of this group in the same way as in the proof of Proposition 2.7 [Na09]. $\square$

Corollary 2.52. *Let W be a benign E-B-pair of rank n, then R_W^{cris} is formally smooth of dimension $\frac{(n-1)n}{2}[K : \mathbb{Q}_p] + 1$.*

Proof. This follows from Lemma 2.48 and

$$\dim_E(D_{\mathrm{dR}}(\mathrm{ad}(W))/\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathrm{ad}(W))) = \frac{(n-1)n}{2}[K : \mathbb{Q}_p],$$

which follows from the condition (2) in Definition 2.44. $\square$

Definition 2.53. Let W be a benign E-B-pair of rank n such that $W|_{G_{K_m}}$ is crystalline. For any $\tau \in \mathfrak{S}_n$, we define a rank one saturated crystabelline E-B-pair $W'_\tau \subseteq W$ such that $D_{\mathrm{cris}}^{K_m}(W'_\tau) = K_0 \otimes_{\mathbb{Q}_p} E e_{\tau(n)} \subseteq D_{\mathrm{cris}}^{K_m}(W)$ and define a sub functor $D_{W,\tau}^{\mathrm{cris}}$ of D_W by

$$D_{W,\tau}^{\mathrm{cris}}(A) := \{[W_A] \in D_W(A) |\ \text{there exists a rank one crystabelline saturated sub } A\text{-}B\text{-pair } W'_A \subseteq W_A \text{ such that } W'_A \otimes_A E = W'_\tau\}.$$

Lemma 2.54. *Under the above condition. The functor $D_{W,\tau}^{\mathrm{cris}}$ is pro-representable by a quotient $R_{W,\tau}^{\mathrm{cris}}$ of R_W which is formally smooth and it's dimension is equal to $n(n-1)[K : \mathbb{Q}_p] + 1$.*

Proof. The proof of relatively representability of $D_{W,\tau}^{\mathrm{cris}} \hookrightarrow D_W$ and the proof of formally smoothness are easily followed from the combination of proofs of Proposition 2.36 and Proposition 2.38 and Lemma 2.51. Here, we only prove dimension

formula. Let $\mathrm{ad}_\tau(W) := \{f \in \mathrm{ad}(W) | f(W'_\tau) \subseteq W'_\tau\}$, then we have the following short exact sequence,

$$0 \to \mathrm{Hom}(W/W'_\tau, W) \to \mathrm{ad}_\tau(W) \to \mathrm{ad}(W'_\tau) \to 0.$$

Taking long exact sequence and by Proposition 2.9, we have the following short exact sequence

$$0 \to \mathrm{H}^1(G_K, \mathrm{Hom}(W/W'_\tau, W)) \to \mathrm{H}^1(G_K, \mathrm{ad}_\tau(W)) \to \mathrm{H}^1(G_K, \mathrm{ad}(W'_\tau)) \to 0.$$

We define $\mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W))$ by the inverse image of $\mathrm{H}^1_f(G_K, \mathrm{ad}(W'_\tau))$ in $\mathrm{H}^1(G_K, \mathrm{ad}_\tau(W))$. Hence we obtain a short exact sequence

$$0 \to \mathrm{H}^1(G_K, \mathrm{Hom}(W/W'_\tau, W)) \to \mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W)) \to \mathrm{H}^1_f(G_K, \mathrm{ad}(W'_\tau)) \to 0.$$

In the same way as in Lemma 2.51, we can show that the natural map $D_W(E[\varepsilon]) \to \mathrm{H}^1(G_K, \mathrm{ad}(W))$ induces an isomorphism

$$D^{\mathrm{cris}}_{W,\tau}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W)).$$

By Theorem 2.8 and Proposition 2.9, we can calculate that

$$\dim_E \mathrm{H}^1(G_K, \mathrm{Hom}(W/W'_\tau, W)) = n(n-1)[K : \mathbb{Q}_p].$$

Because $\mathrm{ad}(W'_\tau) = B_E$ is trivial, hence we have $\mathrm{H}^1_f(G_K, \mathrm{ad}(W'_\tau)) = 1$ by Proposition 2.7 of [Na09]. Hence we have

$$\dim_E \mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W)) = n(n-1)[K : \mathbb{Q}_p] + 1.$$

This proves that $R^{\mathrm{cris}}_{W,\tau}$ is of dimension $n(n-1)[K : \mathbb{Q}_p] + 1$.

□

Lemma 2.55. *Let W be a benign E-B-pair of rank n. Let W_A be a deformation of W over A which is potentially crystalline, then $[W_A] \in D_{W,\mathcal{T}_\tau}(A)$ and $[W_A] \in D^{\mathrm{cris}}_{W,\tau}(A)$ for any $\tau \in \mathfrak{S}_n$.*

Proof. Let W_A be as above. If $W|_{G_{K_m}}$ is crystalline, then $W_A|_{G_{K_m}}$ is crystalline by the proof of Lemma 2.51. Hence it suffices to show that we can write by $D^{K_m}_{\mathrm{cris}}(W_A) \xrightarrow{\sim} K_0 \otimes_{\mathbb{Q}_p} Ae_1 \oplus K_0 \otimes_{\mathbb{Q}_p} Ae_2 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ae_n$ such that $K_0 \otimes_{\mathbb{Q}_p} Ae_i$ is preserved by (φ, G_m) and $\varphi^f(e_i) = \tilde{\alpha}_i e_i$ for a lift $\tilde{\alpha}_i \in A^\times$ of $\alpha_i \in E^\times$ for any $1 \leqq i \leqq n$. For proving this claim, first we note that $D^{K_m}_{\mathrm{cris}}(W_A)$ is a free $K_0 \otimes_{\mathbb{Q}_p} A$-module of rank n and $D^{K_m}_{\mathrm{cris}}(W_A) \otimes_A E \xrightarrow{\sim} D^{K_m}_{\mathrm{cris}}(W)$ by Proposition 1.3.4 and Proposition 1.3.5 [Ki09] and then, for any $\sigma : K_0 \hookrightarrow E$, we can write $D^{K_m}_{\mathrm{cris}}(W_A)_\sigma = Ae_{1,\sigma} \oplus \cdots \oplus Ae_{n,\sigma}$ such that $\varphi^f(e_{i,\sigma}) \equiv \alpha_i e_{i,\sigma} \pmod{\mathfrak{m}_A}$ for any $1 \leqq i \leqq n$. By an easy linear algebra, we can take an A-basis $e'_{1,\sigma}, e'_{2,\sigma}, \cdots, e'_{n,\sigma}$ of $D^{K_m}_{\mathrm{cris}}(W_A)_\sigma$ such that $\varphi^f(e'_{i,\sigma}) = \tilde{\alpha}_i e'_{i,\sigma}$ for a lift $\tilde{\alpha}_i \in A^\times$ of α_i for any i. Because φ and G_m commute and $\alpha_i \neq \alpha_j$, hence $Ae'_{i,\sigma}$ is preserved by G_m. If we take $e_i := e'_{i,\sigma} + \varphi(e'_{i,\sigma}) + \cdots \varphi^{f-1}(e'_{i,\sigma}) \in D^{K_m}_{\mathrm{cris}}(W_A)$, then we have $D^{K_m}_{\mathrm{cris}}(W_A) = K_0 \otimes_{\mathbb{Q}_p} Ae_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ae_n$ satisfying the property of the claim.

□

Lemma 2.56. *Let W be a benign E-B-pair of rank n and $\tau \in \mathfrak{S}_n$ such that $W|_{G_{K_m}}$ is crystalline. Let $[W_A] \in D_{W,\mathcal{T}_\tau}(A)$ be a trianguline deformation over A with a lifting of triangulation $0 \subseteq W_{1,A} \subseteq W_{2,A} \subseteq \cdots W_{n,A} = W_A$. If $W_{i,A}/W_{i-1,A}$ is Hodge-Tate for any $1 \leqq i \leqq n$, then $W_A|_{G_{K_m}}$ is crystalline.*

Proof. First, we prove that $(W_{i,A}/W_{i-1,A})|_{G_{K_m}}$ is crystalline with Hodge-Tate weight $\{k_{i,\sigma}\}_{\sigma\in\mathcal{P}}$. Because $W_{i,A}/W_{i-1,A}$ is written as successive extensions of $W_{\tau,i}/W_{\tau,i-1}$, $W_{i,A}/W_{i-1,A}$ has Hodge-Tate weight $\{k_{i,\sigma}\}_{\sigma\in\mathcal{P}}$. Twisting W_A by a crystalline character $\delta^{-1}_{\alpha_{\tau(i)}} \prod_{\sigma\in\mathcal{P}} \sigma^{-k_{i,\sigma}} : K^\times \to A^\times$, we may assume that $W_{i,A}/W_{i-1,A}$ is an étale Hodge-Tate A-B-pair of rank one with Hodge-Tate weight zero and is a deformation of an étale potentially unramified E-B-pair $W(\tilde{\delta}_{\tau(i)})$. By Sen's theorem ([Se73] or Proposition 5.24 of [Be02]), $W_{i,A}/W_{i-1,A}$ is potentially unramified, hence there exists a unitary homomorphism $\delta : K^\times \to A^\times$ such that $\delta|_{\mathcal{O}_K^\times}$ has a finite image and $W_{i,A}/W_{i-1,A} \xrightarrow{\sim} W(\delta)$ and δ is a lift of $\tilde{\delta}_{\tau(i)}$. Because $(1+\mathfrak{m}_A) \cap A^\times_{\mathrm{torsion}} = \{1\}$, we have $\delta|_{\mathcal{O}_K^\times} = \tilde{\delta}_{\tau(i)}|_{\mathcal{O}_K^\times} : \mathcal{O}_K^\times \to A^\times$, hence $W_{i,A}/W_{i-1,A}|_{G_{K_m}}$ is crystalline. Next, we prove that $W_A|_{G_{K_m}}$ is crystalline by the induction on the rank of W. When $n=1$, we just have proved this. Assume that the lemma is proved for the rank $n-1$ case, then $W_{n-1,A}|_{G_{K_m}}$ is crystalline. If we put $W_A/W_{n-1,A} \xrightarrow{\sim} W(\delta_{A,n})$, then we have $[W_A] \in \mathrm{H}^1(G_K, W_{n-1,A}(\delta_{A,n}^{-1}))$. By considering Hodge-Tate weight of W_A and the condition (3) of Definition 2.44, we obtain $\mathrm{Fil}^0 D_{\mathrm{dR}}(W_{n-1,A}(\delta^{-1}_{A,n})) = 0$. Comparing dimensions and by Proposition 2.7 of [Na09], we have $\mathrm{H}^1_f(G_K, W_{A,n-1}(\delta^{-1}_{A,n})) = \mathrm{H}^1(G_K, W_{A,n-1}(\delta^{-1}_{A,n}))$. Hence $W_A|_{G_{K_m}}$ is crystalline.

□

Lemma 2.57. *Let W be a benign E-B-pair of rank n and W_1 be a rank one crystabelline sub E-B-pair of W, then the saturation $W_1^{sat} := (W_{1,e}^{sat}, W_{1,\mathrm{dR}}^{+,sat})$ of W_1 in W is crystabelline and the natural map $\mathrm{Hom}_{G_K}(W_1^{sat}, W) \to \mathrm{Hom}_{G_K}(W_1, W)$ induced by the natural inclusion $W_1 \hookrightarrow W_1^{sat}$ is isomorphism between one dimensional E-vector spaces.*

Proof. Because $W_{1,e} = W_{1,e}^{sat}$ by Lemma 1.14 of [Na09], so W_1^{sat} is crystabelline. By the definition of benign E-B-pairs, the Hodge-Tate weight of W_1^{sat} is $\{k_{1,\sigma}\}_{\sigma\in\mathcal{P}}$. Consider the following short exact sequence of complexes of G_K-modules defined in p.890 of [Na09]

$$0 \to \mathrm{C}^\bullet(W \otimes (W_1^{sat})^\vee) \to \mathrm{C}^\bullet(W \otimes W_1^\vee) \to ((W\otimes W_1^\vee)^+_{\mathrm{dR}}/(W\otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}})[0] \to 0,$$

where, for an E-B-pair W, we denote by $C^\bullet(W)$ the mapping cone of the natural map $W_e \oplus W^+_{\mathrm{dR}} \to W_{\mathrm{dR}} : (x,y) \mapsto x-y$. From this, we obtain an exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(G_K, W\otimes (W_1^{sat})^\vee) &\to \mathrm{H}^0(G_K, W\otimes W_1^\vee) \\ &\to \mathrm{H}^0(G_K, (W\otimes W_1^\vee)^+_{\mathrm{dR}}/(W\otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}}) \to \cdots \end{aligned}$$

By the condition (3) in Definition 2.44, we have $\dim_E \mathrm{H}^0(G_K, (W \otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}}) = \dim_E \mathrm{H}^0(G_K, (W \otimes W_1^\vee)^+_{\mathrm{dR}}) = n[K:\mathbb{Q}_p]$, Hence, by a usual argument of B^+_{dR}-representations, then we have $\mathrm{H}^0(G_K, (W \otimes W_1^\vee)^+_{\mathrm{dR}}/(W \otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}}) = 0$. Hence the map $\mathrm{H}^0(G_K, W \otimes (W_1^{sat})^\vee)) \xrightarrow{\sim} \mathrm{H}^0(G_K, W \otimes (W_1)^\vee)$ is isomorphism. Finally, for the dimension, we have $\dim_E \mathrm{H}^0(G_K, W \otimes (W_1^{sat})^\vee) = 1$ by the condition (1) in Definition 2.44 and by Proposition 2.9 of [Na09].

□

Lemma 2.58. *Let W be a benign E-B-pair and let W_A be a deformation of W over A. If there exists a rank one crystabelline sub A-B-pair $W_{1,A} \subseteq W_A$ such that the reduction map $W_1 := W_{1,A} \otimes_A E \hookrightarrow W_A \otimes_A E$ remains injective, then the saturation $W_{1,A}^{sat}$ of $W_{1,A}$ in W_A as an E-B-pair is a crystabelline A-B-pair and $W_A/W_{1,A}^{sat}$ is an A-B-pair and $W_{1,A}^{sat} \otimes_A E \xrightarrow{\sim} W_1^{sat} (\subseteq W)$.*

Proof. First, by Proposition 2.14 of [Na09], there exists $\{l_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\leqq 0}$ such that $W_1^{sat} \xrightarrow{\sim} W_1 \otimes W(\prod_{\sigma \in \mathcal{P}} \sigma^{l_\sigma})$. We claim that the inclusion

$$\mathrm{Hom}_{G_K}(W_{1,A} \otimes W_A(\prod_{\sigma \in \mathcal{P}} \sigma^{l_\sigma}), W_A) \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A)$$

induced by the natural inclusion $W_{1,A} \hookrightarrow W_{1,A} \otimes W_A(\prod_{\sigma \in \mathcal{P}} \sigma^{l_\sigma})$ is isomorphism and that these groups are rank one free A-modules. By the same argument as in Lemma 2.57, the cokernel of $\mathrm{Hom}_{G_K}(W_{1,A} \otimes W_A(\prod_{\sigma \in \mathcal{P}} \sigma^{l_\sigma}), W_A) \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A)$ is contained in $\mathrm{H}^0(G_K, (W_A \otimes W_{1,A}^\vee)^+_{\mathrm{dR}}/(W_A \otimes W_{1,A}^\vee \otimes W_A(\prod_{\sigma \in \mathcal{P}} \sigma^{-l_\sigma}))^+_{\mathrm{dR}})$. This is zero by the proof of Lemma 2.57. Hence the natural inclusion

$$\mathrm{Hom}_{G_K}(W_{1,A} \otimes W_A(\prod_{\sigma \in \mathcal{P}} \sigma^{l_\sigma}), W_A) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(W_{1,A}, W_A)$$

is isomorphism. Next, we prove that $\mathrm{Hom}_{G_K}(W_{1,A}, W_A)$ is a free A-module of rank one by the induction on the length of A. When $A = E$, this claim is proved in Lemma 2.57. Assume that A is of length n and assume that the claim is proved for $W_{A'} := W \otimes_A A'$ for a small extension $A \to A'$. We denote by I the kernel of $A \to A'$, $W_{1,A'} := W_{1,A} \otimes_A A'$. From the exact sequence

$$0 \to I \otimes_E \mathrm{Hom}_{G_K}(W_1, W) \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A) \to \mathrm{Hom}_{G_K}(W_{1,A'}, W_{A'})$$

and the induction hypothesis, we have $\mathrm{length}\mathrm{Hom}_{G_K}(W_{1,A}, W_A) \leqq \mathrm{length}A$. On the other hand, the fact that the given inclusion $\iota : W_{1,A} \hookrightarrow W_A$ remains injective after tensoring E and the fact that $\dim_E \mathrm{Hom}_{G_K}(W_1, W) = 1$ and the induction hypothesis imply that the map $A \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A) : a \mapsto a \cdot \iota$ is injection. Hence we have $\mathrm{length}(A) = \mathrm{length}\mathrm{Hom}_{G_K}(W_{1,A}, W_A)$. These facts prove the claim for A. From this claim, the given inclusion $\iota : W_{1,A} \hookrightarrow W_A$ factors through a map

$$\tilde{\iota} : W'_{1,A} := W_{1,A} \otimes W_A(\prod_{\sigma} \sigma^{l_\sigma}) \to W_A$$

and this map is injection because the injection of morphisms of B-pair determined only by the injection of W_e-part of B-pairs and $W_{1,A,e} = (W_{1,A} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma}))_e$. Under this situation, we claim that this $\tilde{\iota}$ gives an isomorphism $W_{1,A}^{sat} \xrightarrow{\sim} W'_{1,A}$ and claim that this satisfies all the properties in this lemma. By the induction on the length of A, we may assume that this claim is proved for A'. First, we prove that $W_A/W'_{1,A}$ is an E-B-pair. For proving this, by Lemma 2.1.4 of [Be08], it suffices to show that $W^+_{A,\mathrm{dR}}/W'^+_{1,A,\mathrm{dR}}$ is a free B^+_{dR}-module. By the snake lemma, we have the following short exact sequence

$$0 \to I \otimes_E W^+_{\mathrm{dR}}/W^{sat+}_{1,\mathrm{dR}} \to W^+_{A,\mathrm{dR}}/W'^+_{1,A,\mathrm{dR}} \to W^+_{A',\mathrm{dR}}/(W'^+_{1,A,\mathrm{dR}} \otimes_A A') \to 0.$$

From this and the induction hypothesis, $W^+_{A,\mathrm{dR}}/W'^+_{1,A,\mathrm{dR}}$ is a free B^+_{dR}-module. Finally, we prove the A-flatness of $W_A/W'_{1,A}$. This follows from the fact that the map $\tilde{\iota}\otimes \mathrm{id}_E : W'_{1,A}\otimes_A E \hookrightarrow W_A \otimes_A E$ is saturated (we can see this from the proof of the above first claim). Hence $W_A/W'_{1,A}$ is an A-B-pair. We finish the proof of this lemma.

□

Lemma 2.59. *Let W be a benign E-B-pair of rank n. For any $\tau \in \mathfrak{S}_n$, we have*

$$D_{W,\mathcal{T}_\tau}(E[\epsilon]) + D^{\mathrm{cris}}_{W,\tau}(E[\varepsilon]) = D_W(E[\varepsilon]).$$

Proof. By Corollary 2.49 and Lemma 2.51 and Lemma 2.54, we have

$$\dim_E D_W(E[\varepsilon]) + \dim_E D^{\mathrm{cris}}_W(E[\varepsilon]) = \dim_E D_{W,\mathcal{T}_\tau}(E[\varepsilon]) + \dim_E D^{\mathrm{cris}}_{W,\tau}(E[\varepsilon]).$$

Hence it suffices to show that

$$D_{W,\mathcal{T}_\tau}(A) \cap D^{\mathrm{cris}}_{W,\tau}(A) = D^{\mathrm{cris}}_W(A)$$

for any $A \in \mathcal{C}_E$. First, the fact that $D^{\mathrm{cris}}_W(A) \subseteq D_{W,\mathcal{T}_\tau}(A) \cap D_{W,\tau}(A)$ follows from Lemma 2.55.

We prove that $D_{W,\mathcal{T}_\tau}(A) \cap D^{\mathrm{cris}}_{W,\tau}(A) \subseteq D^{\mathrm{cris}}_W(A)$ by the induction on the rank n of W. When $n = 1$, this is trivial. Let W be rank n and assume that the lemma is proved for the rank $n-1$ case. Let $[W_A] \in D_{W,\mathcal{T}_\tau}(A) \cap D^{\mathrm{cris}}_{W,\tau}(A)$, then, by the definition of $D_{W,\mathcal{T}_\tau}$ and $D^{\mathrm{cris}}_{W,\tau}$, there exists an A-triangulation $0 \subseteq W_{1,A} \subseteq W_{2,A} \subseteq \cdots \subseteq W_{n-1,A} \subseteq W_{n,A} = W_A$ such that $W_{i,A}\otimes_A E \xrightarrow{\sim} W_{\tau,i}$ for any i and there exists a saturated crystabelline rank one A-B-pair $W'_{1,A} \subseteq W_A$ such that $W'_{1,A}\otimes_A E \xrightarrow{\sim} W'_\tau$. First, we claim that the composition of $W'_{1,A} \hookrightarrow W_A$ and $W_A \to W_A/W_{1,A}$ is injection. Because $\mathrm{Ker}(W'_{1,A} \to W_A/W_{1,A})$ is a sub E-B-pair of $W_{1,A}$ and because we have $\mathrm{Hom}_{G_K}(\mathrm{Ker}(W'_{1,A} \to W_A/W_{1,A}), W_{1,A}) = 0$ by the condition (1) of Definition 2.44 and by Proposition 2.14 of [Na09], hence $W'_{1,A} \to W_A/W_{1,A}$ is injection. Hence, by Lemma 2.58, the saturation $(W'_{1,A})^{sat}$ of $W'_{1,A}$ in $W_A/W_{1,A}$ is a rank one crystabelline A-B-pair satisfying the similar conditions as those of $W'_{1,A} \hookrightarrow W_A$. Hence, by the induction hypothesis, $W_A/W_{1,A}$ is crystabelline and, by the condition (3) of Definition 2.44, Hodge-Tate weight

of $W_A/W_{1,A}$ is $\{k_{2,\sigma}, k_{3,\sigma}, \cdots, k_{n,\sigma}\}_{\sigma\in\mathcal{P}}$ (with multiplicity $[A:E]$). Moreover, by (3) of Definition 2.44, $W'_{1,A}$ has Hodge-Tate weight $\{k_{1,\sigma}\}_{\sigma\in\mathcal{P}}$ (with multiplicity). Because $k_{1,\sigma}\neq k_{i,\sigma}$ for any $i\neq 1$ and there is no extension between different Hodge-Tate weight objects by a theorem of Tate, these imply that W_A is a Hodge-Tate E-B-pair. Hence, by Lemma 2.56, W_A is crystabelline. Hence we have that $[W_A]\in D_W^{\mathrm{cris}}(A)$.

□

The following is the main theorem of § 2, which is a crucial theorem for the applications to Zariski density. This theorem was first discovered by Chenevier (Theorem 3.19 of [Ch09b]) for the $\mathbb{Q}_p$ case.

Definition 2.60. For $R = R_W$ or $R_{W,\mathcal{T}_\tau}$, we denote by

$$t(R) := \mathrm{Hom}_E(\mathfrak{m}_R/\mathfrak{m}_R^2, E)$$

the tangent space of R. We have a natural inclusion $t(R_{W,\mathcal{T}_\tau}) \hookrightarrow t(R_W)$ for any $\tau\in\mathfrak{S}_n$.

Theorem 2.61. *Let W be a benign E-B-pair of rank n. We have an equality*

$$\sum_{\tau\in\mathfrak{S}_n} t(R_{W,\mathcal{T}_\tau}) = t(R_W).$$

Proof. We prove this by the induction on n. When $n=1$, then the theorem is trivial. Assume that the theorem is true for the rank $n-1$ case. Let W be a rank n benign E-B-pair. We take an element $\tau\in\mathfrak{S}_n$. We define a sub functor $D_{W,\tau}$ of D_W by

$D_{W,\tau}(A) := \{[W_A]\in D_W(A)|$ there exists a rank one sub A-B-pair $W_{1,A}\subseteq W_A$ such that the quotient $W_A/W_{1,A}$ is an A-B-pair and $W_{1,A}\otimes_A E \xrightarrow{\sim} W'_\tau\}$,

where W'_τ is defined in Definition 2.53. $D^{\mathrm{cris}}_{W,\tau}$ is a sub functor of $D_{W,\tau}$ and we can show in the same way that

$$D_{W,\tau}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}_\tau(W)),$$

where

$$\mathrm{ad}_\tau(W) := \{f\in\mathrm{ad}(W)|f(W'_\tau)\subseteq W'_\tau\}.$$

Hence, by Lemma 2.59, we have

$$\mathrm{H}^1(G_K,\mathrm{ad}_\tau(W)) + \mathrm{H}^1(G_K,\mathrm{ad}_{\mathcal{T}_\tau}(W)) = \mathrm{H}^1(G_K,\mathrm{ad}(W)).$$

Because, for any $\tau'\in\mathfrak{S}_{\tau,n} := \{\tau'\in\mathfrak{S}_n|\tau'(1)=\tau(n)\}$, we have a natural inclusion $D_{W,\mathcal{T}_{\tau'}}\subseteq D_{W,\tau}$, hence we have natural maps $\mathrm{H}^1(G_K,\mathrm{ad}_{\mathcal{T}_{\tau'}}(W))\to\mathrm{H}^1(G_K,\mathrm{ad}_\tau(W))$. Hence, it suffices to prove that the map

$$\oplus_{\tau'\in\mathfrak{S}_{\tau,n}}\mathrm{H}^1(G_K,\mathrm{ad}_{\mathcal{T}_{\tau'}}(W))\to\mathrm{H}^1(G_K,\mathrm{ad}_\tau(W))$$

is surjection. We prove this surjection as follows. First, by the definition, we have the following short exact sequences of E-B-pairs for any $\tau' \in \mathfrak{S}_{\tau,n}$,

$$0 \to \mathrm{Hom}(W/W'_{\tau}, W) \to \mathrm{ad}_{\tau}(W) \to \mathrm{ad}(W'_{\tau}) \to 0, \tag{1}$$

$$0 \to \{f \in \mathrm{ad}_{\mathcal{T}_{\tau'}}(W) | f(W'_{\tau}) = 0\} \to \mathrm{ad}_{\mathcal{T}_{\tau'}}(W) \to \mathrm{ad}(W'_{\tau}) \to 0. \tag{2}$$

Moreover, we have

$$0 \to \mathrm{Hom}(W/W'_{\tau}, W'_{\tau}) \to \mathrm{Hom}(W/W'_{\tau}, W) \to \mathrm{ad}(W/W'_{\tau}) \to 0, \tag{3}$$

$$0 \to \mathrm{Hom}(W/W'_{\tau}, W'_{\tau}) \to \{f \in \mathrm{ad}_{\mathcal{T}_{\tau'}}(W) | f(W'_{\tau}) = 0\} \to \mathrm{ad}_{\mathcal{T}_{\bar{\tau}'}}(W/W'_{\tau}) \to 0. \tag{4}$$

Here, for any $\tau' \in \mathfrak{S}_{\tau,n}$, we denote by $\mathcal{T}_{\bar{\tau}'} : 0 \subseteq W_{\tau',2}/W'_{\tau} \subseteq W_{\tau',3}/W'_{\tau} \subseteq \cdots \subseteq W_{\tau',n-1}/W'_{\tau} \subseteq W/W'_{\tau}$ the induced triangulation from $\mathcal{T}_{\tau'}$ on W/W'_{τ}. We have $\mathrm{H}^2(G_K, \mathrm{Hom}(W/W'_{\tau}, W'_{\tau})) = 0$ by the condition (1) of Definition 2.44 and Proposition 2.9. We have $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}_{\tau}}(W)) = 0$ from the proof of Proposition 2.40. Hence, from the short exact sequence (4) above, we have $\mathrm{H}^2(G_K, \{f \in \mathrm{ad}_{\tau'}(W) | f(W'_{\tau}) = 0\}) = 0$. From this and from (1) and (2) above, for proving the surjection of $\oplus_{\tau' \in \mathfrak{S}_{\tau,n}} \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}_{\tau'}}(W)) \to \mathrm{H}^1(G_K, \mathrm{ad}_{\tau}(W))$, it suffices to prove the surjection of $\oplus_{\tau' \in \mathfrak{S}_{\tau,n}} \mathrm{H}^1(G_K, \{f \in \mathrm{ad}_{\tau'}(W) | f(W'_{\tau}) = 0\}) \to \mathrm{H}^1(G_K, \mathrm{Hom}(W/W'_{\tau}, W))$. By (3) and (4) above and by $\mathrm{H}^2(G_K, \mathrm{Hom}(W/W'_{\tau}, W'_{\tau})) = 0$, this surjection follows from the surjection of $\oplus_{\tau' \in \mathfrak{S}_{\tau,n}} \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}_{\bar{\tau}'}}(W/W'_{\tau})) \to \mathrm{H}^1(G_K, \mathrm{ad}(W/W'_{\tau}))$, which is the induction hypothesis. Hence we have finished the proof of this theorem

□

3. Construction of p-adic families of two dimensional trianguline representations.

In this section, we generalize Kisin's construction of p-adic families of two dimensional trianguline representations for any p-adic field. In the next section, generalizing Kisin's method in [Ki10], we use these families to prove Zariski density of two dimensional crystalline representations for any p-adic field.

3.1. **Almost $\mathbb{C}_p$-representations.** In this subsection, we recall some rings of Lubin-Tate's p-adic periods defined by Colmez ([Co02]) and the definition of almost $\mathbb{C}_p$-representations defined by Fontaine ([Fo03]). Using these, we prove some propositions concerning to Banach G_K-modules which we need for the construction of p-adic families of trianguline representations.

Let π_K be a fixed uniformizer of K. $P(X) \in \mathcal{O}_K[X]$ be a monic polynomial of degree $q := p^f$ such that $P(X) \equiv \pi_K X$ (mod deg 2) and $P(X) \equiv X^q$ (mod π_K). From $P(X)$, we obtain the Lubin-Tate formal group law $\mathcal{F}_{\pi_K}$ of $\mathcal{O}_K$ on which the multiplication by π_K is given by $[\pi_K] = P(X)$. Let $\chi_{\mathrm{LT}} : G_K \to \mathcal{O}_K^{\times}$ be the Lubin-Tate character associated to π_K. Let $A_{\mathrm{inf},K} := \mathbb{A}^+ \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K$. This ring is equipped with the weak topology on which G_K acts continuously. $A_{\mathrm{inf},K}$ also has a $\mathcal{O}_K$-linear continuous $\varphi_K := \varphi^f$-action. Any element of $A_{\mathrm{inf},K}$ can be written uniquely

of the form $\sum_{k=0}^{\infty}[x_k]\pi_K^k$ ($x_k \in \tilde{\mathbb{E}}^+$). We put $B_{\mathrm{inf},K} := A_{\mathrm{inf},K}[p^{-1}]$. By Lemma 8.3 of [Co02], for any element $x \in \tilde{\mathbb{E}}^+$, there exists unique element $\{x\} \in A_{\mathrm{inf},K}$ such that $\{x\}$ is a lift of x and $\varphi_K(\{x\}) = [\pi_K](\{x\})(= P(\{x\}))$. We fix $\{\omega_n\}_{n\geqq 0}$ such that $\omega_1 \in \mathfrak{m}_{\overline{K}}$ is a primitive $[\pi_K]$-torsion point of $\mathcal{F}_{\pi_K}$ and $[\pi_K](\omega_{n+1}) = \omega_n$ for any $n \geqq 0$, then $(\bar{\omega}_n)_{n\geqq 0}$ defines an element in $\tilde{\mathbb{E}}^+ \xrightarrow{\sim} \varprojlim_n \mathcal{O}_{\overline{K}}/\pi_K\mathcal{O}_{\overline{K}}$ where the projective limit is given by q-th power Frobenius map. We define $\omega_K := \{(\bar{\omega}_n)_{n\geqq 0}\} \in A_{\mathrm{inf},K}$. By the definition of $\{-\}$ and by the uniqueness of $\{-\}$, the actions of G_K and φ_K on ω_K are given by $g(\omega_K) = [\chi_{\mathrm{LT}}(g)](\omega_K)$ for any $g \in G_K$ (which converges for the weak topology) and by $\varphi_K(\omega_K) = [\pi_K](\omega_K)$. We put $\tilde{\pi}_K := (\bar{\pi}_n)_{n\geqq 0} \in \tilde{\mathbb{E}}^+$ where $\pi_n \in \mathcal{O}_{\overline{K}}$ satisfies that $\pi_0 = \pi_K$ and $\pi_{n+1}^q = \pi_n$ for any n, then we define $A_{\max,K} := \widehat{A_{\mathrm{inf},K}[\frac{[\tilde{\pi}_K]}{\pi_K}]}$ the p-adic completion of $A_{\mathrm{inf},K}[\frac{[\tilde{\pi}_K]}{\pi_K}]$. We define $B^+_{\max,K} = A_{\max,K}[p^{-1}]$, this is a K-Banach space with continuous actions of G_K and φ_K. By the definition, we have a canonical isomorphism $K \otimes_{K_0} B^+_{\max,\mathbb{Q}_p} \xrightarrow{\sim} B^+_{\max,K}$ (Remark 7.13 of [Co02]). By Lemma 8.8 and Proposition 8.9 of [Co02], there exists a power series $F_K(X) \in K[[X]]$ (Lubin-Tate's logarithm) such that $F_K(X) \circ [a] = aF_K(X)$ for any $a \in \mathcal{O}_K$ and $t_K := F_K(\omega)$ converges in $A_{\max,K}$ such that $\varphi_K(t_K) = \pi_K t_K, g(t_K) = \chi_{\mathrm{LT}}(g)t_K$ for any $g \in G_K$. We define $B_{\max,K} := B^+_{\max,K}[t_K^{-1}]$. We define $B^+_{\mathrm{dR}} := \varprojlim_n B^+_{\mathrm{inf},K}/(\mathrm{Ker}(\theta))^n$ equipped with the projective limit topology of K-Banach spaces $B^+_{\mathrm{inf},K}/(\mathrm{Ker}(\theta))^n$ whose $\mathcal{O}_K$-lattice is defined as the image of $A_{\mathrm{inf},K} \to B^+_{\mathrm{inf},K}/(\mathrm{Ker}(\theta))^n$. By Proposition 7.12 of [Co02], this B^+_{dR} is canonically topologically isomorphic to the usual B^+_{dR}. We define $B_{\mathrm{dR}} := B^+_{\mathrm{dR}}[t^{-1}] = B^+_{\mathrm{dR}}[t_K^{-1}]$.

Under this situation, we define a functor from the category of φ_K-modules to the category of almost $\mathbb{C}_p$-representations defined by Fontaine. We can see this construction as a very special case of a generalization of Berger's results ([Be09]) to the case of Lubin-Tate period rings.

Definition 3.1. We say that D is a φ_K-module over K if D is a finite dimensional K-vector space with a K-linear isomorphism $\varphi_K : D \xrightarrow{\sim} D$.

Let D be a φ_K-module over K, we extend the action of φ_K to $\hat{K}^{\mathrm{ur}} \otimes_K D$ by $\varphi_K(a \otimes x) := \varphi_K(a) \otimes \varphi_K(x)$, where $\hat{K}^{\mathrm{ur}}$ is the p-adic completion of the maximal unramified extension K^{ur} of K and $\varphi_K \in \mathrm{Gal}(K^{\mathrm{ur}}/K)$ is the lift of q-th Frobenius in $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$. The Deudonné-Manin theorem gives a decomposition $\hat{K}^{\mathrm{ur}} \otimes_K D = \oplus_{s\in\mathbb{Q}} D_s$, where for any $s = \frac{a}{h} \in \mathbb{Q}$ such that $(a,h) \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ are co-prime, D_s is zero or a finite direct sum of $D_{a,h} := \hat{K}^{\mathrm{ur}}e_1 \oplus \hat{K}^{\mathrm{ur}}e_2 \oplus \cdots \oplus \hat{K}^{\mathrm{ur}}e_h$ such that $\varphi_K(e_1) = e_2, \varphi_K(e_2) = e_3, \cdots, \varphi_K(e_{h-1}) = e_h, \varphi_K(e_h) = \pi_K^a e_1$. We define the set of slopes of D as the set of $s \in \mathbb{Q}$ such that $D_s \neq 0$. We define a $\hat{K}^{\mathrm{ur}}$-semi-linear G_K-action on $\hat{K}^{\mathrm{ur}} \otimes_K D$ by $g(a \otimes x) := g(a) \otimes x$ for any $g \in G_K$, $a \in \hat{K}^{\mathrm{ur}}$, $x \in D$, then D_s is preserved by this G_K-action for any $s \in \mathbb{Q}$ because the actions of G_K and φ_K commute each other. For any $s = \frac{a}{h}$, if we define $D'_s := \{x \in D_s | \varphi_K^h(x) = \pi_K^a x\}$,

then we have $D_s = \hat{K}^{\mathrm{ur}} \otimes_{K_h^{\mathrm{ur}}} D'_s$ and D'_s is preserved by G_K and φ_K, where K_h^{ur} is the unramified extension of K of degree h.

The notion of almost $\mathbb{C}_p$-representations was defined by Fontaine ([Fo03]).

Definition 3.2. Let W be a $\mathbb{Q}_p$-Banach space equipped with a continuous $\mathbb{Q}_p$-linear G_K-action. We say that W is an almost $\mathbb{C}_p$-representation if there exists $\mathbb{Q}_p$-representations V_1, V_2 and an integer $d \in \mathbb{Z}_{\geqq 0}$ such that V_1 is a sub G_K-module of W and V_2 is a sub G_K-module of $\mathbb{C}_p^d$ and there exists an isomorphism $W/V_1 \xrightarrow{\sim} \mathbb{C}_p^d/V_2$ as $\mathbb{Q}_p$-Banach G_K-modules.

Remark 3.3. By Theorem C of [Fo03], $B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}$ is an almost $\mathbb{C}_p$-representation for any $k \geqq 0$. By Theorem B of [Fo03], for any continuous $\mathbb{Q}_p$-linear G_K-morphism $f : W_1 \to W_2$ between almost $\mathbb{C}_p$-representations W_1, W_2, it is known that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ (as $\mathbb{Q}_p[G_K]$-modules) are almost $\mathbb{C}_p$-representations and $\mathrm{Im}(f)$ is also an almost $\mathbb{C}_p$-representation which is a closed subspace of W_2.

Let D be a φ_K-module over K. We will prove that $X_0(D) := (B^+_{\max,K} \otimes_K D)^{\varphi_K=1}$ is an almost $\mathbb{C}_p$-representation.

Lemma 3.4. *Let D be a φ_K-module over K.*

(1) *$X_0(D)$ is an almost $\mathbb{C}_p$-representation.*

(2) *If any slope s of D satisfies $s > 0$, then $X_0(D) = 0$.*

Proof. The proof is similar to that of Proposition 2.2 of [Be09]. If we denote by $\hat{K}^{\mathrm{ur}} \otimes_K D = \oplus_{s \in \mathbb{Q}} D_s$ as above, then we have $B^+_{\max,K} \otimes_K D = \oplus_{s\in\mathbb{Q}} B^+_{\max,K} \otimes_{\hat{K}^{\mathrm{ur}}} D_s$ as a φ_K-module and, for any $s = \frac{a}{h}$, $B^+_{\max,K} \otimes_{\hat{K}^{\mathrm{ur}}} D_s = B^+_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s$ is preserved by the actions of G_K and φ_K. Hence, it suffices to show that, for any $s = \frac{a}{h}$, $(B^+_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s)^{\varphi_K=1}$ is an almost $\mathbb{C}_p$-representation and is zero if $a > 0$. By the definition of D'_s, we have a canonical inclusion $(B^+_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s)^{\varphi_K=1} \subseteq B^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s$. By 8.5 of [Co02], for $a > 0$ we have $B^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K} = 0$ and, for $a \leqq 0$ we have a short exact sequence

$$0 \to K_h^{\mathrm{ur}} t_{K_h^{\mathrm{ur}}}^{-a} \to B^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K} \to B^+_{\mathrm{dR}}/t^{-a} B^+_{\mathrm{dR}} \to 0,$$

where $t_{K_h^{\mathrm{ur}}} \in B^+_{\max,K} = B^+_{\max,K_h^{\mathrm{ur}}}$ is defined from $(K_h^{\mathrm{ur}}, \pi_K, \varphi_K^h)$ in the same way as in the definition of t_K defined from (K, π_K, φ_K). Moreover, because $B^+_{\mathrm{dR}}/t^{-a} B^+_{\mathrm{dR}} \otimes_{K_h^{\mathrm{ur}}} D'_s$ is a B^+_{dR}-representation, so this is also an almost $\mathbb{C}_p$-representation by Theorem 5.13 of [Fo03]. Hence, $B^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s$ is also an almost $\mathbb{C}_p$-representation. Because $(B^+_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s)^{\varphi_K=1} = \mathrm{Ker}(\varphi_K - 1 : B^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s \to B^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s)$, then $(B^+_{\max,K} \otimes_{K_h^{\mathrm{ur}}} D'_s)^{\varphi_K=1}$ is also an almost $\mathbb{C}_p$-representation by Remark 3.3. □

As an application of this lemma, we obtain the following corollary. We fix an embedding $\sigma : K \hookrightarrow E$. For a K-vector space M and an E-vector space N, we

denote by $M \otimes_{K,\sigma} N$ the tensor product of M and N over K, where we view N as a K-vector space by using $\sigma : K \hookrightarrow E$.

Corollary 3.5. *Let $\alpha \in E^\times$ be a non-zero element, then $(B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha}$ is an almost $\mathbb{C}_p$-representation and, for any positive integer k such that $k > e_K v_p(\alpha)$, the natural map $(B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}) \otimes_{K,\sigma} E$ is injection, where e_K is the absolute ramified index of K. Moreover, if we denote the cokernel of this inclusion by U_k, we have the following short exact sequence of E-Banach almost $\mathbb{C}_p$-representations,*

$$0 \to (B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}) \otimes_{K,\sigma} E \to U_k \to 0.$$

Proof. For $\alpha \in E^\times$, we define a φ_K-module D_α over K by $D_\alpha := Ee$ such that $\varphi_K(ae) = \alpha^{-1}ae$ for any $a \in E$. D_α has unique slope $-e_K v_p(\alpha)$ and we have a natural isomorphism

$$X_0(D_\alpha) \xrightarrow{\sim} (B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha}.$$

Hence, by Lemma 3.4, $(B^+_{\max,K} \otimes_K E)^{\varphi=\alpha}$ is a non-zero almost $\mathbb{C}_p$-representation. Moreover, by using Proposition 8.10 of [Co02], we can show that, for any $k \geqq 0$, we have an equality

$$(B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \cap (t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} E) = (t^k_K B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha}.$$

This is isomorphic to $X_0(D_{\alpha\sigma(\pi_K)^{-k}})$ where $D_{\alpha\sigma(\pi_K)^{-k}} = Ee$ is defined by $\varphi_K(e) = \alpha^{-1}\sigma(\pi_K)^k e$. Because, $D_{\alpha\sigma(\pi_K)^{-k}}$ has unique slope $(k - e_K v_p(\alpha))$, so we have $X_0(D_{\alpha\sigma(\pi_K)^{-k}}) = 0$ when $k > e_K v_p(\alpha)$ by Lemma 3.4. This implies that the natural map $(B^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}) \otimes_{K,\sigma} E$ is injection. Because both of these are almost $\mathbb{C}_p$-representations, hence the cokernel U_k is also an almost $\mathbb{C}_p$-representation by Remark 3.3 in particular U_k is an E-Banach space. □

For two K-Banach spaces M_1 and M_2, we denote by $M_1 \hat{\otimes}_K M_2$ the complete tensor product of M_1 and M_2 over K. Let R be a complete topological E-algebra. We say that R is an E-Banach algebra if there exists a map $|-|_R : R \to \mathbb{R}_{\geqq}$ which satisfies, for any $x, y \in R$, $a \in E$,

(1) $|1|_R = 1$, $|x|_R = 0$ if and only if $x = 0$,
(2) $|x+y|_R \leqq \max\{|x|_R, |y|_R\}$,
(3) $|xy|_R \leqq |x|_R |y|_R$ and $|ax|_R = |a|_p |x|_R$

and if the topology of R is defined by the metric induced from $|-|_R$.

Lemma 3.6. *Let R be an E-Banach algebra and $\alpha \in R$ be an element of R such that $\alpha - 1$ is topologically nilpotent, then there exists $u \in (\hat{K}^{\mathrm{ur}} \hat{\otimes}_{K,\sigma} R)^\times$ such that $\varphi_K(u) = \alpha u$.*

Proof. The proof is same as that of Lemma 3.6 of [Ki03]. □

Here, we recall some terminologies concerning to Banach modules from §2 of [Bu07]. Let R be an E-Banach algebra and let M be a topological R-module. We say that M is a Banach R-module if M is a complete topological R-module with a map $|-|: M \to \mathbb{R}_{\geqq 0}$ satisfying that, for any $m, n \in M$, $a \in R$,

(1) $|m| = 0$ if and only if $m = 0$,
(2) $|m+n| \leqq \max\{|m|, |n|\}$,
(3) $|am| \leqq |a|_R |m|$(where $|-|_R$ is an E-Banach norm on R)

and the topology on M is defined by the metric induced from $|-|$. Let M be a Banach R-module. We say that M is potentially orthonormalizable if there exist a map $|-|: M \to \mathbb{R}_{\geqq 0}$ as above and a subset $\{e_i\}_{i\in I}$ of M such that

(1) for any $m \in M$, there exists unique $\{a_i\}_{i\in I}$ ($a_i \in R$) such that $a_i \to 0 (i \to \infty)$ and that $m = \sum_{i\in I} a_i e_i$,
(2) in the situation (1), we have $|m| = \max_{i\in I}\{|a_i|_R\}$.

We say that a Banach R-module M has property (Pr) if there exists a Banach R-module N such that $M \oplus N$ is potentially orthonormalizable.

The following proposition is also a generalization of Corollary 3.7 of [Ki03] which will play a crucial role in the next subsection.

Proposition 3.7. *Let R be an E-Banach algebra and let $Y \in R^{\times}$ be a unit of R. We assume that there exists a finite Galois extension E' of E and there exists $\lambda \in (R \otimes_E E')^{\times}$ such that $E'[\lambda] \subseteq R \otimes_E E'$ is an étale E'-algebra and that $Y\lambda^{-1} - 1$ is topologically nilpotent in $R \otimes_E E'$, then, for sufficient large $k \in \mathbb{Z}_{>0}$, there exists a Banach R-module U_k which has property (Pr) with a continuous R-linear G_K-action such that there exists a G_K-equivariant short exact sequence of Banach R-modules with property (Pr) ,*

$$0 \to (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R \to U_k \to 0$$

Proof. If we decompose $E'[\lambda] = \prod_{i\in I} E_i \subseteq R \otimes_E E'$ such that each E_i is a finite extension of E' and if we denote by $\lambda_i \in E_i$ the image of λ in E_i, then we can decompose $R \otimes_E E' = \prod_{i\in I} R_i$ such that $E_i \subseteq R_i$. By Corollary 3.5, for any $k \in \mathbb{Z}_{>0}$ such that $k > e_K v_p(\lambda_i)$ for any $i \in I$, we have a short exact sequence of E_i-Banach spaces

$$0 \to (B^+_{\mathrm{max},K} \otimes_{K,\sigma} E_i)^{\varphi_K = \lambda_i} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} E_i \to U_{k,i} \to 0.$$

Hence, if we take the complete tensor product over E_i of this sequence with R_i, for any $i \in I$, we obtain a short exact sequence of potentially orthonormalizable R_i-Banach spaces

$$0 \to (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R_i)^{\varphi_K = \lambda_i} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R_i \to U_{k,i} \hat{\otimes}_{E_i} R_i \to 0.$$

By the assumption, the element $Y\lambda_i^{-1} - 1$ is topologically nilpotent in R_i, hence by Lemma 3.6, we have an element $u_i \in (\hat{K}^{\mathrm{ur}} \hat{\otimes}_{K,\sigma} R_i)^{\times}$ such that $\varphi_K(u_i) = Y\lambda_i^{-1} u_i$.

Multiplying u_i to the above short exact sequence, we obtain a short exact sequence

$$0 \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R_i)^{\varphi_K=Y} \to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R_i \to (U_{k,i}\hat{\otimes}_{E_i}R_i) \to 0.$$

Summing up for all $i \in I$, we obtain

$$0 \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}(R\otimes_E E'))^{\varphi_K=Y} \to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(R\otimes_E E') \to \oplus_{i\in L}(U_{k,i}\hat{\otimes}_{E_i}R_i) \to 0.$$

Finally, by taking $\mathrm{Gal}(E'/E)$-fixed part, if we put $U_k := (\oplus_{i\in I}(U_{k,i}\hat{\otimes}_{E_i}R_i))^{\mathrm{Gal}(E'/E)}$ (this has property (Pr)), we obtain a short exact sequence

$$0 \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} \to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to U_k \to 0$$

satisfying all the conditions in the proposition.

□

Let V be an E-representation, then we define

$$D^+_{\mathrm{cris}}(V) := (B^+_{\max}\otimes_{\mathbb{Q}_p}V)^{G_K},\ \mathrm{Fil}^0D_{\mathrm{cris}}(V) := D_{\mathrm{cris}}(V)\cap \mathrm{Fil}^0D_{\mathrm{dR}}(V),$$

where we write $B^+_{\max} := B^+_{\max,\mathbb{Q}_p}$. Then, we have a natural inclusion $D^+_{\mathrm{cris}}(V) \subseteq \mathrm{Fil}^0D_{\mathrm{cris}}(V)$, which is not equal in general.

Lemma 3.8. *Let $\alpha \in E^\times$ be a non zero element. If a φ-sub module D of $D_{\mathrm{cris}}(V)^{\varphi^f=\alpha}$ is contained in $\mathrm{Fil}^0D_{\mathrm{dR}}(V)$, then D is also contained in $D^+_{\mathrm{cris}}(V)^{\varphi^f=\alpha}$.*

Proof. It suffices to show that if an element $x \in (B_{\max}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\alpha}$ satisfies that $\varphi^i(x) \in B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E$ for any $i \in \mathbb{Z}_{\geqq 0}$, then $x \in (B^+_{\max}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\alpha}$. If we write $x = \frac{a}{t^n}$ for some $a \in (B^+_{\max}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\alpha p^{fn}}$ and $n \geqq 0$, then we have $\frac{\varphi^i(a)}{p^{ni}t^n} = \varphi^i(x) \in B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E$ for any $0 \leqq i \leqq f-1$. Hence, we have

$$\varphi^i(a) \in (B^+_{\max}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\alpha p^{nf}} \cap t^nB^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E = (t^n_{K_0}B^+_{\max}\otimes_{\mathbb{Q}_p}E)^{\varphi^f=\alpha p^{nf}},$$

where the last equality follows from Proposition 8.10 of [Co02]. Hence, we can write $a = \varphi^{-i}(t^n_{K_0})a_i$ for some $a_i \in B^+_{\max}\otimes_{\mathbb{Q}_p}E$ for any $1 \leqq i \leqq f-1$ and then we can write by $a = (\prod_{i=0}^{f-1}\varphi^{-i}(t^n_{K_0}))a'$ for some $a' \in B^+_{\max}\otimes_{\mathbb{Q}_p}E$ by Lemma 8.18 of [Co02]. Because $\prod_{i=0}^{f-1}\varphi^{-i}(t_{K_0}) \in K_0^\times t$ by Lemma 8.17 of [Co02], we have $x = \frac{a}{t^n} \in B^+_{\max}\otimes_{\mathbb{Q}_p}E$.

□

3.2. **Construction of the finite slope subspace: for general p-adic field case.** In this subsection, we generalize Kisin's construction of X_{fs} for any p-adic field. If we admit Proposition 3.7, the construction and the proof is almost same as in the $\mathbb{Q}_p$-case, the only difference is that we need to consider all the embeddings $\sigma : K \hookrightarrow E$. But, for the convenience of readers and the author, here we reprove this construction in full detail.

Let X be a separated rigid analytic space (in the sense of Tate) over E. Let M be a free $\mathcal{O}_X$-module of rank d for some $d \geqq 1$ equipped with a continuous $\mathcal{O}_X$-linear G_K-action, where "continuous" means that, for any admissible open

affinoid $U = \mathrm{Spm}(R)$ of X, the action of G_K on $\Gamma(U, M)$ is continuous on which the topology is defined as the direct sum topology of R. We denote by $M^\vee := \mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X)$ the $\mathcal{O}_X$-dual of M. For any point $x \in X$, we denote the local ring at x by $\mathcal{O}_{X,x}$, the maximal ideal at x by $\mathfrak{m}_x$, the residue field at x by $E(x)$ which is a finite extension of E and we denote by $M(x)$ the fiber of M at x which is a d-dimensional $E(x)$-representation of G_K. By Lemma 3.18 of [Ch09a], we can take an affinoid covering $\{U_i\}_{i\in I}$ of X such that $M_i := \Gamma(U_i, M)$ is a free $R_i := \Gamma(U_i, \mathcal{O}_X)$-module which has a G_K-stable finite free R_i^0-module M_i^0 such that $M_i = M_i^0[p^{-1}]$ for a model $R_i^0 \subseteq R_i$, where, for an affinoid A, a model is defined as a topologically finite generated complete $\mathcal{O}_E$-sub algebra of A which generates A after inverting p. Then, we can apply the theory of [Be-Co08] of families of p-adic representations to M_i (and M_i^0) for any $i \in I$. By [Be-Co08], there exists a unique monic polynomial $P_{M_i}(T) \in K \otimes_{\mathbb{Q}_p} R_i[T]$ of dimension d , which is the characteristic polynomial of Sen's operator on $D^L_{\mathrm{Sen}}(M_i)$ of Proposition 4.1.2 of [Be-Co08] (where L is a sufficiently large finite extension of K), such that for any point $x \in \mathrm{Spm}(R_i)$, the specialization of $P_{M_i}(T)$ at x gives the Sen's polynomial $P_{M(x)}(T) \in K \otimes_{\mathbb{Q}_p} E(x)[T]$ of $M(x)$. By the uniqueness of $D^L_{\mathrm{Sen}}(M_i)$, $\{P_{M_i}(T)\}_{i\in I}$ glue together to a monic polynomial $P_M(T) \in K \otimes_{\mathbb{Q}_p} \mathcal{O}_X[T]$. By the canonical decomposition $K \otimes_{\mathbb{Q}_p} E = \oplus_{\sigma\in\mathcal{P}} E : a \otimes b \mapsto (\sigma(a)b)_\sigma$, $P_M(T)$ decomposes into the σ-components $P_M(T) = (P_M(T)_\sigma)_{\sigma\in\mathcal{P}} \in \oplus_{\sigma\in\mathcal{P}} \mathcal{O}_X[T]$.

Now we assume that the constant term of $P_M(T)_\sigma$ is zero for any $\sigma \in \mathcal{P}$. We denote by $P_M(T)_\sigma = TQ_\sigma(T)$ for some $Q_\sigma(T) \in \mathcal{O}_X[T]$.

Before stating the theorem, we recall some terminologies of rigid geometry (from § 5 of [Ki03]) which we need for the construction of X_{fs}.

Let $X = \mathrm{Spm}(R)$ be an affinoid variety over E and U be an admissible open in X. We say that U is scheme theoretically dense in X if there exists a Zariski open $V \subseteq \mathrm{Spec}(R)$ which is dense in $\mathrm{Spec}(R)$ for the Zariski topology and $U = V^{\mathrm{an}}$ where V^{an} is the admissible open set of X associated to V. For any rigid analytic variety X over E and an admissible open U of X, we say that U is scheme theoretically dense in X if there exists an admissible affinoid covering $\{U_i\}_{i\in I}$ of X such that $U \cap U_i$ is scheme theoretically dense in U_i for any $i \in I$. The typical example is the following. Let $f \in \mathcal{O}_X$ be any element, then we denote $X_f := \{x \in X | f(x) \neq 0\}$ which is an admissible open in X. If f is a non-zero divisor, then X_f is scheme theoretically dense in X.

Next, let $Y \in \mathcal{O}_X^\times$ be an invertible function on X. We say that a morphism $f : \mathrm{Spm}(R) \to X$ is Y-small if there exists a finite extension E' of E and $\lambda \in (R \otimes_E E')^\times$ such that $E'[\lambda] \subseteq R \otimes_E E'$ is a finite étale E'-algebra and that $Y\lambda^{-1} - 1 \in R \otimes_E E'$ is topologically nilpotent. A typical example of Y-small morphism is following. For any $x \in X$ and $n \in \mathbb{Z}_{\geqq 1}$, the natural map $f : \mathrm{Spm}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n) \to X$ is Y-small.

The following theorem is the key theorem for the construction of p-adic families of trianguline representations, which is exactly the generalization of Proposition

5.4 of [Ki03] for any K-case. For an E-affinoid R, we denote by $B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}R := \varprojlim_k B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}R$ equipped with the projective limit topology.

Theorem 3.9. *Let X be a separated rigid analytic variety over E, M be a free $\mathcal{O}_X$-module of rank d with a continuous $\mathcal{O}_X$-linear G_K-action. Let $Y \in \mathcal{O}_X^{\times}$ be an invertible function. We assume that the constant term of $P_M(T)_\sigma$ is zero for any $\sigma \in \mathcal{P}$. Then, there exists unique Zariski closed sub space X_{fs} of X satisfying the following conditions.*

(1) *For any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, $X_{fs,Q_\sigma(i)}$ is scheme theoretically dense in X_{fs}.*
(2) *For any Y-small map $f : \mathrm{Spm}(R) \to X$ which factors through $X_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, the following two conditions are equivalent.*
   (i) *$f : \mathrm{Spm}(R) \to X$ factors through X_{fs}.*
   (ii) *Any R-linear G_K-equivariant map $h : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}R$ factors through $h' : M^\vee \otimes_{\mathcal{O}_X} R \to K \otimes_{K_0} (B^+_{\max}\hat{\otimes}_{\mathbb{Q}_p}R)^{\varphi^f = Y}$.*

The proof of this theorem is almost same as that of Kisin's in the $\mathbb{Q}_p$-case. But for convenience of readers and of the author, we reprove this in detail. As in [Ki03], we prove this theorem by several steps. First we prove the following.

Lemma 3.10. *Let X, M be as above. Let X' be a separated rigid analytic variety over E and $f : X' \to X$ be a flat E-morphism. If there exists a Zariski closed sub space $X_{fs} \subseteq X$ which satisfies (1) and (2) of the above theorem, then $X'_{fs} := X_{fs} \times_X X' \subseteq X'$ also satisfies (1) and (2) for X' and $M' := f^*M$ and $Y' := f^*(Y) \in \mathcal{O}_{X'}^{\times}$.*

Proof. First, the condition (1) is satisfied by X'_{fs} because the notion of scheme theoretically dense is preserved by flat base changes and because we have $f^*(P_M(T)) = P_{f^*M}(T)$. That X'_{fs} satisfies (2) is trivial. □

Next, we prove uniqueness of X_{fs}.

Lemma 3.11. *If two Zariski closed sub varieties X_1 and X_2 of X satisfy the conditions (1) and (2), then $X_1 = X_2$.*

Proof. By the above Lemma 3.10, for any affinoid open $U \subseteq X$ and for any $i = 1, 2$, $X_i \cap U \subseteq U$ satisfies (1) and (2) for U because the inclusion $U \hookrightarrow X$ is flat. Hence, for an admissible covering $\{U_i\}_{i\in I}$ of X, it suffices to prove that $X_1 \cap U_i = X_2 \cap U_i$ for any $i \in I$. Hence we may assume that $X = \mathrm{Spm}(R)$ is an affinoid. We denote by $X_1 = \mathrm{Spm}(R/I_1)$, $X_2 = \mathrm{Spm}(R/I_2)$ for some ideals $I_1, I_2 \subseteq R$. If we denote $X_3 := \mathrm{Spm}(R/I_1 \cap I_2)$, then we claim that X_3 also satisfies the conditions (1) and (2) . For (1), by the assumption, we have inclusions $R/I_j \hookrightarrow R/I_j[\frac{1}{Q_\sigma(i)}]$ for any $j = 1, 2$ and for any $\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}$, hence we have an inclusion $R/I_1 \cap I_2 \hookrightarrow R/I_1 \cap I_2[\frac{1}{Q_\sigma(i)}]$ for any $\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}$, which proves that X_3 satisfies (1). For proving that X_3 satisfies (2), we take a Y-small morphism $f : \mathrm{Spm}(R') \to X$ which factors $f : \mathrm{Spm}(R') \to X_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}$. We denote by

$Y' := f^*(Y) \in R'^{\times}$. If f satisfies (ii) of (2), then by the definition of X_1 and X_2, f factors through X_1 and X_2, hence f also factors through X_3 because $X_1, X_2 \subseteq X_3$. Next we assume that f satisfies (i) of (2) . We have a canonical decomposition

$$\begin{aligned} K \otimes_{K_0} (B^+_{\max} \hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y'} &= ((K \otimes_{K_0} B^+_{\max}) \hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y'} \\ &= (B^+_{\max,K} \hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y'} = \oplus_{\sigma \in \mathcal{P}} (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'} \end{aligned}$$

and $B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} R' = \oplus_{\sigma \in \mathcal{P}} B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'$. Hence, it suffices to show that, for any $\sigma \in \mathcal{P}$, any G_K-equivariant R'-linear map $h : M^{\vee} \otimes_{\mathcal{O}_X} R' \to B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'$ factors through $M^{\vee} \otimes_{\mathcal{O}_X} R' \to (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'}$. Because $Q_{\sigma}(i)$ is invertible in R' for any $\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}$ by the assumption, the natural map

$$(B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} M \otimes_{\mathcal{O}_X} R')^{G_K} \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} M \otimes_{\mathcal{O}_X} R')^{G_K}$$

is isomorphism for any $\sigma \in \mathcal{P}$ and any $k \in \mathbb{Z}_{\geqq 1}$ by Proposition 2.5 of [Ki03]. Hence, it suffices to show that for some $k \in \mathbb{Z}_{\geqq 1}$, any G_K-equivariant map $h : M^{\vee} \otimes_{\mathcal{O}_X} R' \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R')$ factors through $M^{\vee} \otimes_{\mathcal{O}_X} R' \to (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'}$. We choose a $k \in \mathbb{Z}_{\geqq 1}$ such that there exists a short exact sequence of Banach R'-modules with property (Pr) as in Proposition 3.7 whose exactness is preserved by any complete tensor base change,

$$0 \to (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R' \to U_{k,\sigma} \to 0.$$

Moreover, if we denote by $\mathrm{Spm}(R'_i) := f^{-1}(X_i) \subseteq \mathrm{Spm}(R')$ for $i = 1, 2$, then we have an inclusion $R' \hookrightarrow R'_1 \oplus R'_2$ because f factors through X_3. From these, the above short exact sequence can be embedded in the following short exact sequence

$$0 \to \oplus_{i=1}^2 (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R'_i)^{\varphi_K = Y'} \to \oplus_{i=1}^2 B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'_i \to \oplus_{i=1}^2 U_{k,\sigma} \hat{\otimes}_{R'} R'_i \to 0.$$

Then, the composition of h with $B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R' \hookrightarrow \oplus_{i=1}^2 B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'_i$ factors through $M^{\vee} \otimes_{\mathcal{O}_X} R' \hookrightarrow \oplus_{i=1}^2 (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R'_i)^{\varphi_K = Y'}$ by the definition of X_i. Hence, by the above diagram of the two short exact sequences, h also factors through $M^{\vee} \otimes_{\mathcal{O}_X} R' \to (B^+_{\max,K} \hat{\otimes}_{K,\sigma} R'_i)^{\varphi_K = Y'}$. Hence, X_3 also satisfies (1) and (2).

Hence, for proving the lemma, we may assume that $X_1 \subseteq X_2$. We put $W \subseteq X_2$ the support of I_1/I_2 with the reduced structure. If $x \in X_2$ satisfies $Q_{\sigma}(i)(x) \neq 0$ for any $\sigma \in \mathcal{P}$, $i \in \mathbb{Z}_{\leqq 0}$, then, for any $n \geqq 1$, the natural Y-small map $\mathrm{Spm}(\mathcal{O}_{X_2,x}/\mathfrak{m}_x^n) \to X_2$ factors through $\mathrm{Spm}(\mathcal{O}_{X_2,x}/\mathfrak{m}_x^n) \to X_1$ by the definition on X_1 and X_2. This implies that there exists a map $\mathcal{O}_{X_1,x} \to \hat{\mathcal{O}}_{X_2,x}$ such that the composition of this with the natural map $\mathcal{O}_{X_2,x} \to \mathcal{O}_{X_1,x}$ is the natural map $\mathcal{O}_{X_2,x} \to \hat{\mathcal{O}}_{X_2,x}$. This implies that the natural quotient map $\mathcal{O}_{X_2,x} \to \mathcal{O}_{X_1,x}$ is isomorphism, hence we have $x \notin W$. Hence, we have an inclusion $W \subseteq \cup_{\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}} \{x \in X_2 | Q_{\sigma}(i)(x) = 0\}$. By Lemma 5.7 [Ki03], then there exists a $Q \in \mathcal{O}_{X_2}$ a finite product of $Q_{\sigma}(i)$ such that $X_{2,Q} \subseteq X_2 \setminus W = X_1 \setminus W \subseteq X_1 \subseteq X_2$. Then, the condition (1) for X_2 implies that $X_1 = X_2$. We finish to prove the lemma.

$\square$

Let $\{U_i\}_{i\in I}$ be an admissible affinoid covering of X such that $U_{i,fs} \subseteq U_i$ exists for all $i \in I$. By the uniqueness of X_{fs}, $U_{i,fs}$ glue to a Zariski closed sub space $X'_{fs} \subseteq X$ satisfying that $X'_{fs} \cap U_i = U_{i,fs}$ for all $i \in I$.

Lemma 3.12. *In the above situation, $X'_{fs} \subseteq X$ satisfies the conditions* (1) *and* (2) *in the theorem.*

Proof. That X'_{fs} satisfies (1) is trivial. We show that X'_{fs} satisfies (2). Let $f : \mathrm{Spm}(R) \to X$ be a Y-small map which factors through $X_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. Because X is separated, $f^{-1}(U_i) := \mathrm{Spm}(R_i)$ is also an affinoid for any $i \in I$. First we show that (i) implies (ii). We assume that f factors through X'_{fs}. Let $h : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R$ be a R-linear G_K-equivariant map. By Proposition 2.5 of [Ki03], it suffices to show that $h : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y}$ for some $k \in \mathbb{Z}_{\geqq 1}$. We choose a $k \in \mathbb{Z}_{\geqq 1}$ such that there exists a short exact sequence of Banach R-modules with property (Pr) as in Proposition 3.7

$$0 \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R \to U_{k,\sigma} \to 0.$$

By the property (Pr), this short exact sequence can be embedded into the following

$$0 \to \prod_{i\in I}(B^+_{\max,K}\hat{\otimes}_{K,\sigma} R_i)^{\varphi_K = Y} \to \prod_{i\in I} B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R_i \to \prod_{i\in I} U_{k,\sigma}\hat{\otimes}_R R_i \to 0.$$

By the assumption, $h_i : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R_i$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma} R_i)^{\varphi_K = Y}$ for any $i \in I$. Hence, $h : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R$ also factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y}$, which can be seen from the above diagram of the two exact sequences.

Next, we assume that, for any $\sigma \in \mathcal{P}$, any R-linear G_K-equivariant map $h : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y}$. Because $Q_\sigma(j) \in R^\times$ for any $\sigma \in \mathcal{P}$ and any $j \in \mathbb{Z}_{\leqq 0}$, we have isomorphisms $(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} M \otimes_{\mathcal{O}_X} R)^{G_K} \otimes_R R_i \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} M \otimes_{\mathcal{O}_X} R_i)^{G_K}$ for any $k \geqq 1$ and for any $i \in I$ by Corollary 2.6 of [Ki03]. Hence, any R_i-linear G_K-equivariant maps $h_i : M^\vee \otimes_{\mathcal{O}_X} R \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R_i$ factor through $M^\vee \otimes_{\mathcal{O}_X} R \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma} R_i)^{\varphi_K = Y}$ for any $i \in I$. This implies that $f|_{\mathrm{Spm}(R_i)} : \mathrm{Spm}(R_i) \to U_i$ factors through $U_{i,fs}$ for any $i \in I$. Hence, f also factors through X'_{fs}. $\square$

By this lemma, it suffices to construct X_{fs} for a sufficiently small affinoid open $X = \mathrm{Spm}(R)$. We may assume that $|Y|$ satisfies $|Y||Y^{-1}| < \frac{1}{|\pi_K|_p}$, where $|-| : R \to \mathbb{R}_{\geqq 0}$ is a map which define the topology of R as in §3.2. Then, we construct $X_{fs} \subseteq \mathrm{Spm}(R)$ as follows. First, we construct an ideal of R which determines X_{fs}. Let $\lambda \in \overline{E}$ be any element such that $|Y^{-1}|^{-1} \leqq |\lambda|_p \leqq |Y|$ and E' be a finite Galois extension of E which contains λ. By Corollary 3.5, we can take a sufficiently large

$k \in \mathbb{Z}_{\geqq 1}$ such that, for any λ as above and for any $\sigma \in \mathcal{P}$, there exists a short exact sequence of E'-Banach spaces

$$0 \to (B^{+}_{\mathrm{max},K}\hat{\otimes}_{K,\sigma}E')^{\varphi_K=\sigma(\pi_K)\lambda} \to B^{+}_{\mathrm{dR}}/t^k B^{+}_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}E' \to U_{k,\sigma,\lambda} \to 0.$$

For any $x \in \tilde{\mathbb{E}}^{+}$ such that $v(x) > 0$, we define an element

$$P(x, \frac{Y}{\sigma(\pi_K)\lambda}) := \sum_{n\in\mathbb{Z}} \varphi_K^n([x])(\frac{Y}{\sigma(\pi_K)\lambda})^n \in (B^{+}_{\mathrm{max},K}\hat{\otimes}_{K,\sigma}R\otimes_E E')^{\varphi_K=\frac{\sigma(\pi_K)\lambda}{Y}}.$$

This element converges because

$$|\frac{\sigma(\pi_K)\lambda}{Y}| \leqq |\sigma(\pi_K)|_p|\lambda|_p|Y^{-1}| < |\sigma(\pi_K)|_p|\lambda|_p|Y|^{-1}|\pi_K|_p^{-1} \leqq 1$$

and because $\varphi_K^n([x])(\frac{Y}{\sigma(\pi_K)\lambda})^n \to 0$ ($n \to +\infty$) (see Corollary 4.4 of [Ki03]). For any $\sigma \in \mathcal{P}$ and for any R-linear G_K-equivariant map $h : M^{\vee} \to B^{+}_{\mathrm{dR}}/t^k B^{+}_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$, we consider the composition of this map with

$$B^{+}_{\mathrm{dR}}/t^k B^{+}_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to B^{+}_{\mathrm{dR}}/t^k B^{+}_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R\otimes_E E' : v \mapsto P(x, \frac{Y}{\sigma(\pi_K)\lambda})v$$

and with $B^{+}_{\mathrm{dR}}/t^k B^{+}_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R\otimes_E E' \to U_{k,\sigma,\lambda}\hat{\otimes}_{E'}(R\otimes_E E')$ which is the base change of the surjection $B^{+}_{\mathrm{dR}}/t^k B^{+}_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}E' \to U_{k,\sigma,\lambda}$ in the above short exact sequence. We denote this composition by

$$h_{x,\lambda} : M^{\vee} \to U_{k,\sigma,\lambda}\hat{\otimes}_{E'}(R\otimes_E E').$$

If we fix an orthonormalizable E'-base $\{e_i\}_{i\in I}$ of $U_{k,\sigma,\lambda}$, then, for any $m \in M^{\vee}$, we can write uniquely by

$$h_{x,\lambda}(m) = \sum_{i\in I} a_{x,\lambda,i}(m)e_i \text{ for some } \{a_{x,\lambda,i}(m)\}_{i\in I} \subseteq R\otimes_E E'.$$

Then, we define an ideal

$$I(h, x, \lambda, m) \subseteq R\otimes_E E'$$

which is generated by $a_{x,\lambda,i}(m)$ for all $i \in I$. Because, for any $\tau \in \mathrm{Gal}(E'/E)$, we have $\tau(I(h, x, \lambda, m)) = I(h, x, \tau(\lambda), m) \subseteq R\otimes_E E'$, the ideal

$$\sum_{\tau\in\mathrm{Gal}(E'/E)} I(h, x, \tau(\lambda), m) \subseteq R\otimes_E E'$$

descends to an ideal $I'(h, x, \lambda, m) \subseteq R$ and this ideal is independent of the choice of E'. We define an ideal I of R by

$$I := \sum_{h,x,\lambda,m} I'(h, x, \lambda, m) \subseteq R,$$

where we take all h, x, λ, m and $\sigma \in \mathcal{P}$ as above. Next, we consider the kernels of the natural maps $R/I \to R/I[\frac{1}{\prod_{l=1}^{n} Q_{\sigma_l}(i_l)}]$ for any $n \geqq 1$, $\sigma_l \in \mathcal{P}, i_l \in \mathbb{Z}_{\leqq 0}$. Because R is noetherien, there exists the largest ideal $(I \subseteq)J \subseteq R$ such

that $R/I \to R/I[\frac{1}{\prod_{l=1}^{n} Q_{\sigma_l}(i_l)}]$ factor through $R/I \to R/J$ and the induced maps $R/J \to R/I[\frac{1}{\prod_{l=1}^{n} Q_{\sigma_l}(i_l)}]$ are injection for any n and σ_l and i_l. We claim that $X_{fs} = \mathrm{Spm}(R/J)$, which proves the theorem.

Lemma 3.13. $\mathrm{Spm}(R/J) \subseteq \mathrm{Spm}(R)$ *satisfies the conditions* (1) *and* (2) *in the theorem.*

Proof. Because $R/J \to R/J[\frac{1}{Q_\sigma(i)}]$ are injection for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$ by the definition of J, $\mathrm{Spm}(R/J)$ satisfies the condition (1). We show that $\mathrm{Spm}(R/J)$ satisfies (2). Let $f : \mathrm{Spm}(R') \to \mathrm{Spm}(R)$ be a Y-small map which factors through $\mathrm{Spm}(R') \to \mathrm{Spm}(R)_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. Under this situation, we first prove that (ii) implies (i). We assume that , for any $\sigma \in \mathcal{P}$, any $h : M^\vee \to B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'$ factor through $M^\vee \to (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y}$. Then, for any $h : M^\vee \to B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R$ and $\lambda \in E'$ and $x \in \tilde{\mathbb{E}}^+$ as in the construction of the ideal $I \subseteq R$, the maps

$$P(x, \frac{Y}{\sigma(\pi_K)\lambda}) h \otimes_R \mathrm{id}_{R'} : M^\vee \to U_{k,\sigma,\lambda} \hat{\otimes}_{E'} (R' \otimes_E E')$$

are zero because the multiplication by $P(x, \frac{Y}{\sigma(\pi_K)\lambda})$ sends $(B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R)^{\varphi = Y}$ to $(B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R \otimes_E E')^{\varphi_K = \sigma(\pi_K)\lambda}$. Hence, the map $R \to R'$ factors through $R/I \to R'$ and then, because $Q_\sigma(i) \in R'^{\times}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, $R/I \to R'$ factors through $R/J \to R'$ by the definition of J.

Next, we assume that $f : \mathrm{Spm}(R') \to \mathrm{Spm}(R)$ factors through $\mathrm{Spm}(R') \to \mathrm{Spm}(R/J)$. Let $h : M^\vee \to B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'$ be a R'-linear G_K-equivariant map. We want to show that this map factors through $M^\vee \to (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y}$. By Galois descent, it suffices to show that this factors through $M^\vee \to (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R' \otimes_E E')^{\varphi_K = Y}$ for a sufficiently large finite Galois extension E' of E. Hence, by the definition of Y-smallness, we may assume that there exists $\lambda \in E$ such that $Y\lambda^{-1} - 1$ is topologically nilpotent in R'. Then, we have $|Y^{-1}|^{-1} \leqq |f^*(Y)^{-1}|^{-1}_{R'} = |\lambda|_p = |f^*(Y)|_{R'} \leqq |Y|$, hence λ satisfies the condition in the construction of $I \subseteq R$. By the definition of I, for any $m \in M^\vee$, $P(x, \frac{Y}{\sigma(\pi_K)\lambda}) h(m)$ is an element in $(B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = \sigma(\pi_K)\lambda}$ for any $x \in \tilde{\mathbb{E}}^+$ such that $v(x) > 0$. If we take an element $u \in (\hat{K}^{\mathrm{ur}} \hat{\otimes}_{K,\sigma} R')^{\times, \varphi_K = \frac{\lambda}{Y}}$ as in Lemma 3.6, then we have

$$t_K u h(m) \in (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = \sigma(\pi_K)\lambda}$$

because $t_K u \in (B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = \frac{\sigma(\pi_K)\lambda}{Y}}$ and because the R'-module generated by the sets $\{P(x, \frac{Y}{\sigma(\pi_K)\lambda})\}_{x \in \tilde{\mathbb{E}}^+, v(x) > 0}$ is dense in $(B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = \frac{\sigma(\pi_K)\lambda}{Y}}$, which can be proved in the same way as Corollary 4.6 of [Ki03] by using Lemma 4.3.1 of [Ke05], and because $(B^+_{\mathrm{max},K} \hat{\otimes}_{K,\sigma} R')^{\varphi_K = \sigma(\pi_K)\lambda}$ is closed in $B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} R'$ by

Proposition 3.7. Hence, we have

$$\begin{aligned} uh(m) & \in \tfrac{1}{t_K}(B^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda}\cap B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R' \\ & = (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda}, \end{aligned}$$

where the last equality follows from Proposition 8.10 of [Co02]. Hence, we obtain $h(m)\in (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=Y}$. We finish to prove the lemma, hence we finish the proof or the theorem.

□

We will apply this construction to the rigid analytic space associated to the universal deformation rings of mod p representations of G_K. Before doing this, we reprove some important general properties of X_{fs} proved in Corollary 5.16 of [Ki03] for general K case.

Let $\mathrm{Spm}(R)\subseteq X_{fs}$ be an affinoid open of X_{fs}, we assume that this inclusion is Y-small. By Proposition 3.7, there exists $k>0$ such that, for any $\sigma\in\mathcal{P}$, there exists a short exact sequence of Banach R-modules with property (Pr)

$$0\to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y}\to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R\to U_{k,\sigma}\to 0.$$

We denote by M_R the restriction of M to $\mathrm{Spm}(R)$.

Proposition 3.14. *For any $\sigma\in\mathcal{P}$, let $H_\sigma\subseteq R$ be the smallest ideal of R such that any R-linear G_K-equivariant morphisms $h: M_R^\vee\to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$ factor through $M_R^\vee\to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}H_\sigma$ and put $H:=\prod_{\sigma\in\mathcal{P}}H_\sigma\subseteq R$, then the following hold:*

(1) *The natural maps*

$$((B^+_{\max,K}\hat{\otimes}_{K,\sigma}M_R)^{\varphi_K=Y})^{G_K}\to (B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}M_R)^{G_K}$$

are isomorphism for any $\sigma\in\mathcal{P}$ (i.e. the natural map

$$K\otimes_{K_0}D^+_{\mathrm{cris}}(M_R)^{\varphi_K=Y}\to (B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}M_R)^{G_K}$$

is isomorphism).

(2) $\mathrm{Spm}(R)\setminus V(H)$ *and* $\mathrm{Spm}(R)\setminus V(H_\sigma)$ *(for any $\sigma\in\mathcal{P}$) are scheme theoretically dense in* $\mathrm{Spm}(R)$*, where $V(H_*):=\mathrm{Spm}(R/H_*)$.*

(3) *For any $x\in\mathrm{Spm}(R)$, $M(x)$ is a split trianguline $E(x)$-representation, more precisely, there exists a $\{k_\sigma\}_{\sigma\in\mathcal{P}}\in\prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\geqq 0}$ such that there exists a short exact sequence of $E(x)$-B-pairs*

$$0\to W(\delta_{Y(x)}\prod_{\sigma\in\mathcal{P}}\sigma^{-k_\sigma})\to W(M(x))\to W(\det(M(x))\delta_{Y(x)}^{-1}\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma})\to 0,$$

where, for any $\lambda\in E(x)^\times$, we define a homomorphism $\delta_\lambda: K^\times\to E(x)^\times$ such that $\delta_\lambda(\pi_K)=\lambda$ and $\delta_\lambda|_{\mathcal{O}_K^\times}$ is trivial.

Proof. First, we prove (1). Let's consider a point $x \in \mathrm{Spm}(R)$ such that $x \in \mathrm{Spm}(R)_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and any $i \in \mathbb{Z}_{\leqq 0}$. Then, by the definition of X_{fs}, any G_K-map $h : M_R^\vee \to B_{\mathrm{dR}}^+/t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ factor through $M_R^\vee \to (B_{\max,K}^+ \otimes_{K,\sigma} \mathcal{O}_{X,x}/\mathfrak{m}_x^n)^{\varphi_K = Y}$ for any $n \geqq 1$. We denote by $V \subseteq \mathrm{Spm}(R)$ the set of points satisfying the above condition. By the same argument as in Lemma 3.12, it suffices to show that the natural map $R \to \prod_{x \in V, n \geqq 1} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is injection. Let $f \in R$ be an element in the kernel of this map. Let $W \subseteq \mathrm{Spm}(R)$ be the support of f with the reduced structure, then we have $W \subseteq \cup_{\sigma \in \mathcal{P}, i \leqq 0} V(Q_\sigma(i))$, hence there exists $Q \in R$ a finite product of $Q_\sigma(i)$ such that $W \subseteq V(Q)$ by Lemma 5.7 of [Ki03]. Hence we have $X_Q \subseteq X \setminus W \subseteq X$. This implies that $f = 0 \in R[\frac{1}{Q}]$ and then the condition (1) of Theorem 3.9 implies that $f = 0$ in R. Hence, the map $R \to \prod_{x \in V, n \geqq 1} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is injection.

Next we prove (2). Let $x \in \mathrm{Spm}(R)$ such that $x \in \mathrm{Spm}(R)_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. Then, for any quotient R' of $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ and for any $\sigma \in \mathcal{P}$, we have an isomorphism

$$(B_{\mathrm{dR}}^+/t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} M_R)^{G_K} \otimes_R R' \xrightarrow{\sim} (B_{\mathrm{dR}}^+/t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} (M_R \otimes_R R'))^{G_K}$$

and this is a free R'-module of rank one by Corollary 2.6 of [Ki03]. If $H_\sigma \mathcal{O}_{X,x} \neq \mathcal{O}_{X,x}$, then we have $(B_{\mathrm{dR}}^+/t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} M_R)^{G_K} \otimes_R \mathcal{O}_{X,x}/H_\sigma \mathcal{O}_{X,x} = 0$ by the definition of H_σ, this is a contradiction. Hence, we have $H_\sigma \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ for any σ and for such a point x, hence we also have $H\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$. This implies that $V(H) \subseteq \cup_{\sigma \in \mathcal{P}, i \leqq 0} V(Q_\sigma(i))$. Hence, there exists $Q \in R$ a finite product of $Q_\sigma(i)$ such that $\mathrm{Spm}(R)_Q \subseteq \mathrm{Spm}(R) \setminus V(H) \subseteq \mathrm{Spm}(R)$ by Lemma 5.7 of [Ki03]. Because $\mathrm{Spm}(R)_Q$ is scheme theoretically dense, so $\mathrm{Spm}(R) \setminus V(H)$ is also scheme theoretically dense. Because $V(H) \subseteq V(H_\sigma)$, $\mathrm{Spm}(R) \setminus V(H_\sigma)$ is also scheme theoretically dense.

Finally we prove (3). Let x be any point of $\mathrm{Spm}(R)$. By (2), for any $\sigma \in \mathcal{P}$, there exists $n_\sigma \geqq 0$ such that $H_\sigma \subseteq \mathfrak{m}_x^{n_\sigma}$ and $H_\sigma \not\subseteq \mathfrak{m}_x^{n_\sigma+1}$. By the definition of H_σ, there exists a G_K-map $h : M_R^\vee \to (B_{\max,K}^+ \hat{\otimes}_{K,\sigma} H_\sigma)^{\varphi_K = Y}$ which, by composing with $B_{\max,K}^+ \hat{\otimes}_{K,\sigma} H_\sigma \to B_{\max,K}^+ \otimes_{K,\sigma} \mathfrak{m}_x^{n_\sigma}/\mathfrak{m}_x^{n_\sigma+1}$, induces a nonzero map $M_R^\vee \to (B_{\max,K}^+ \otimes_{K,\sigma} \mathfrak{m}_x^{n_\sigma}/\mathfrak{m}_x^{n_\sigma+1})^{\varphi_K = Y(x)}$. Hence, by taking a suitable $E(x)$-linear projection $\mathfrak{m}_x^{n_\sigma}/\mathfrak{m}_x^{n_\sigma+1} \to E(x)$, we obtain a non zero G_K-map $M_R^\vee \to (B_{\max,K}^+ \otimes_{K,\sigma} E(x))^{\varphi_K = Y(x)}$. This implies that $(B_{\max,K}^+ \otimes_{K,\sigma} M(x))^{\varphi_K = Y(x)} \neq 0$, hence this also implies that $D_{\mathrm{cris}}^+(M(x))^{\varphi_K = Y(x)} \neq 0$, then $M(x)$ is a split trianguline $E(x)$-representation as in the statement of (3). □

3.3. **Construction of p-adic families of two dimensional trianguline representations for any p-adic fields.** In this subsection, we will apply the results of § 3.2 to the rigid analytic space associated with a universal deformation ring of mod p Galois representation.

Let $\mathcal{C}_{\mathcal{O}}$ be the category of local Artin $\mathcal{O}$-algebras with the residue field $\mathbb{F}$. Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous homomorphism, we denote by $\overline{V}$ a two dimensional

$\mathbb{F}$-representation defined by $\bar{\rho}$. As in the case of E-representations, we define a functor $D_{\bar{\rho}} : \mathcal{C}_{\mathcal{O}} \to Sets$ by

$$D_{\bar{\rho}}(A) := \{ \text{ equivalent classes of deformations of } \overline{V} \text{ over } A\}$$

for any $A \in \mathcal{C}_{\mathcal{O}}$. In this paper, for simplicity, we assume that $\overline{V}$ satisfies that

$$\mathrm{H}^0(G_K, \mathrm{ad}(\overline{V})) = \mathbb{F}.$$

Then, $D_{\bar{\rho}}$ is pro-representable by a complete noetherian local $\mathcal{O}$-algebra $R_{\bar{\rho}}$ with the residue field $\mathbb{F}$. $R_{\bar{\rho}} \xrightarrow{\sim} \varprojlim_n R_{\bar{\rho}}/\mathfrak{m}^n$ is equipped with the projective limit topology, where $R_{\bar{\rho}}/\mathfrak{m}^n$ is equipped with the discrete topology for any $n \in \mathbb{Z}_{\geq 1}$. When $\overline{V}$ does not satisfy $\mathrm{H}^0(G_K, \mathrm{ad}(\overline{V})) = \mathbb{F}$, we can prove the same theorems below in the same way if we consider the framed deformations. Let V^{univ} be the universal deformation over $R_{\bar{\rho}}$, which is a rank two free $R_{\bar{\rho}}$-module with a $R_{\bar{\rho}}$-linear continuous G_K-action. Let $\mathfrak{X}(\bar{\rho})$ be the rigid analytic space over E associated to $R_{\bar{\rho}}$. Let $\tilde{V}^{\mathrm{univ}}$ be a free $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-module associated to V^{univ}, which is naturally equipped with an $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-linear continuous G_K-action induced from that on V^{univ}, where "continuous" means that G_K acts continuously on $\Gamma(U, \tilde{V}^{\mathrm{univ}})$ for any affinoid opens $U = \mathrm{Spm}(R) \subseteq \mathfrak{X}(\bar{\rho})$.

Remark 3.15. For a point $x \in \mathfrak{X}(\bar{\rho})$, the fiber V_x of $\tilde{V}^{\mathrm{univ}}$ at x is a two dimensional $E(x)$-representation such that a mod p reduction is isomorphic to $\bar{\rho}$ for a G_K-stable $\mathcal{O}_{E(x)}$-lattice of V_x. Because we assume that $\mathrm{End}_{\mathbb{F}[G_K]}(\bar{\rho}) = \mathbb{F}$, we also have $\mathrm{End}_{E(x)[G_K]}(V_x) = E(x)$ for any $x \in \mathfrak{X}(\bar{\rho})$.

Let $\mathcal{W}_E$ be the rigid analytic space over E which represents the functor $D_{\mathcal{W}_E}$ (from the category of rigid analytic spaces over E to the category of groups) defined by, for any rigid analytic space Y over E,

$$D_{\mathcal{W}_E}(Y) := \{\delta : \mathcal{O}_K^\times \to \Gamma(Y, \mathcal{O}_Y^\times) \text{ continuous homomorphisms } \},$$

where "continuous" is the same meaning as in the definition of $\tilde{V}^{\mathrm{univ}}$. It is known that $\mathcal{W}_E$ is the rigid analytic space associated to the Iwasawa algebra $\mathcal{O}[[\mathcal{O}_K^\times]]$, which is non-canonically isomorphic to a finite (this number is equal to the number of torsion points in $\mathcal{O}_K^\times$) union of $[K : \mathbb{Q}_p]$-dimensional open unit disc over E. We denote by

$$\delta_0^{\mathrm{univ}} : \mathcal{O}_K^\times \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times)$$

the universal continuous homomorphism, which is the composition of the natural maps $\mathcal{O}_K^\times \to \mathcal{O}[[\mathcal{O}_K^\times]]^\times : a \mapsto [a]$ and $\mathcal{O}[[\mathcal{O}_K^\times]]^\times \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times)$. Using a fixed π_K, we extend δ_0^{univ} to $K^\times$ by

$$\delta^{\mathrm{univ}} : K^\times \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times) \text{ such that } \delta^{\mathrm{univ}}(\pi_K) = 1,\ \delta^{\mathrm{univ}}|_{\mathcal{O}_K^\times} = \delta_0^{\mathrm{univ}}.$$

By local class field theory, we can uniquely extend δ^{univ} to a character

$$\tilde{\delta}^{\mathrm{univ}} : G_K^{\mathrm{ab}} \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times) \text{ such that } \delta^{\mathrm{univ}} = \tilde{\delta}^{\mathrm{univ}} \circ \mathrm{rec}_K.$$

We denote by
$$X(\bar{\rho}) := \mathfrak{X}(\bar{\rho}) \times_E \mathcal{W}_E \times_E \mathbb{G}^{an}_{m,E}.$$
Let Y be the canonical parameter of $\mathbb{G}^{an}_{m,E}$. We denote the projections by
$$p_1 : X(\bar{\rho}) \to \mathfrak{X}(\bar{\rho}),\ p_2 : X(\bar{\rho}) \to \mathcal{W}_E,\ p_3 : X(\bar{\rho}) \to \mathbb{G}^{an}_{m,E}.$$
We denote by $N := p_1^*\tilde{V}^{\mathrm{univ}}$ and denote by $M := N((p_2^*\tilde{\delta}^{\mathrm{univ}})^{-1})$. These are rank two free $\mathcal{O}_{X(\bar{\rho})}$-modules with $\mathcal{O}_{X(\bar{\rho})}$-linear continuous G_K-actions. A point x of $X(\bar{\rho})$ can be written as a triple $x = ([V_x], \delta_x, \lambda_x)$ where, V_x is an $E(x)$-representation such that the mod p-reduction of a suitable G_K-stable $\mathcal{O}_{E(x)}$-lattice of V_x is isomorphic to $\bar{V}$ (after scalar extension) and $\delta_x : \mathcal{O}_K^\times \to E(x)^\times$ is a continuous homomorphism and $\lambda_x \in E(x)^\times$ (where $E(x)$ is the residue field of x), i.e. in this paper, if we write $x = ([V_x], \delta_x, \lambda_x)$ then we assume that these are defined over $E(x)$. We denote by
$$P_M(T) = (P_M(T)_\sigma)_{\sigma\in\mathcal{P}} = (T^2 - a_{1,\sigma}T + a_{0,\sigma})_{\sigma\in\mathcal{P}} \in K \otimes_{\mathbb{Q}_p} \mathcal{O}_{X(\bar{\rho})}[T]$$
the Sen's polynomial of M. We denote by $X_0(\bar{\rho}) \subseteq X(\bar{\rho})$ the Zariski closed subspace defined by the ideal generated by $a_{0,\sigma}$ for all $\sigma \in \mathcal{P}$. Let $M_0 := M|_{X_0(\bar{\rho})}$ be the restriction of M to $X_0(\bar{\rho})$, then we have
$$P_{M_0}(T) = (T(T - a_{1,\sigma}))_{\sigma\in\mathcal{P}} \in K \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_0(\bar{\rho})}[T].$$
We denote by $Q_\sigma(T) := T - a_{1,\sigma} \in \mathcal{O}_{X_0(\bar{\rho})}[T]$ for any $\sigma \in \mathcal{P}$. Under this situation, we apply Theorem 3.9 to $X_0(\bar{\rho})$ and M_0 and $Y := (p_3^*Y)|_{X_0(\bar{\rho})}$, then we obtain a Zariski closed subspace
$$\mathcal{E}(\bar{\rho}) := X_0(\bar{\rho})_{fs} \subseteq X_0(\bar{\rho}).$$
For this $\mathcal{E}(\bar{\rho})$, we have a following theorem, which is a modified version of Proposition 10.4 of [Ki03]. For any $\lambda \in \overline{E}^\times$, we define a unramified continuous homomorphism $\delta_\lambda : K^\times \to \overline{E}^\times$ such that $\delta_\lambda(\pi_K) := \lambda$ and $\delta_\lambda|_{\mathcal{O}_K^\times}$ is trivial. For a point $\delta \in \mathcal{W}_E(\overline{E})$, i.e. for a continuous homomorphism $\delta : \mathcal{O}_K^\times \to \overline{E}^\times$, we denote by the same letter $\delta : K^\times \to \overline{E}^\times$ such that $\delta(\pi_K) = 1$ and $\delta|_{\mathcal{O}_K^\times} = \delta$.

Theorem 3.16. *$\mathcal{E}(\bar{\rho})$ has the following properties.*

(1) *For any point $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$, there exists $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 0}$ and, if we put $\delta_1 := \delta_x\delta_{\lambda_x}\prod_{\sigma\in\mathcal{P}}\sigma^{-k_\sigma}$, there exists a short exact sequence of $E(x)$-B-pairs.*
$$0 \to W(\delta_1) \to W(V_x) \to W(\det(V_x)\delta_1^{-1}) \to 0.$$

(2) *Conversely, if a point $x := ([V_x], \delta_x, \lambda_x) \in X(\bar{\rho})$ satisfies the following conditions (i) and (ii), then $x \in \mathcal{E}(\bar{\rho})$.*

(i) *V_x is a split trianguline $E(x)$-representation with a triangulation*
$$\mathcal{T}_x : 0 \subseteq W(\delta_x\delta_{\lambda_x}) \subseteq W(V_x).$$

(ii) *$(V_x, \mathcal{T}_x)$ satisfies all the assumptions in Proposition 2.40.*

Proof. (1) follows from (3) of Proposition 3.14.

We prove (2). Extending the scalar from E to $E(x)$, we may assume that $E(x) = E$. Let $x := ([V_x], \delta_x, \lambda_x) \in X(\bar{\rho})$ be an E-rational point satisfying all the conditions in (2), then the trianguline deformation functor $D_{V_x,\mathcal{T}_x}$ is represented by a formally smooth quotient $R_{V_x,\mathcal{T}_x}$ of the universal deformation ring R_{V_x} of V_x by Proposition 2.40. Moreover, we have a canonical isomorphism $R_{V_x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho}),p_1(x)}$ and then $V_x^{\mathrm{univ}} := \tilde{V}^{\mathrm{univ}} \otimes_{\mathcal{O}_{\mathfrak{X}(\bar{\rho})}} \widehat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho}),p_1(x)}$ is the universal deformation of V_x by Proposition 9.5 of [Ki03]. Taking a quotient, we have a map $\widehat{\mathcal{O}}_{X(\bar{\rho}),x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho}),p_1(x)} \hat{\otimes}_E \widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)} \hat{\otimes}_E \widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)} \to R_{V_x,\mathcal{T}_x} \hat{\otimes}_E \widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)} \hat{\otimes}_E \widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)}$. By the definition of $R_{V_x,\mathcal{T}_x}$, there exists a continuous homomorphism $\delta_{\mathcal{T}_x} : K^{\times} \to R^{\times}_{V_x,\mathcal{T}_x}$ which gives the universal triangulation, i.e. we have the following triangulations of $V_x^{\mathrm{univ}} \otimes_{R_{V_x}} R_{V_x,\mathcal{T}_x}/\mathfrak{m}^n$,

$$\mathcal{T}_{\mathrm{univ},n} : 0 \subseteq W(\delta_{\mathcal{T}_x,n}) \subseteq W(V_x^{\mathrm{univ}} \otimes_{R_{V_x}} R_{V_x,\mathcal{T}_x}/\mathfrak{m}^n)$$

where $\mathfrak{m}$ is the maximal ideal of $R_{V_x,\mathcal{T}_x}$ and $\delta_{\mathcal{T}_x,n}$ is the composition of $\delta_{\mathcal{T}_x}$ with the natural quotient map $R_{V_x,\mathcal{T}_x} \to R_{V_x,\mathcal{T}_x}/\mathfrak{m}^n$ for any $n \geqq 1$. We put $\lambda_{\mathcal{T}_x} := \delta_{\mathcal{T}_x}(\pi_K) \in R^{\times}_{V_x,\mathcal{T}_x}$. On the other hand, we denote the composition of the universal homomorphism $\delta_0^{\mathrm{univ}} : \mathcal{O}_K^{\times} \to \mathcal{O}_{\mathcal{W}_E}^{\times}$ with $\mathcal{O}_{\mathcal{W}_E}^{\times} \to \widehat{\mathcal{O}}^{\times}_{\mathcal{W}_E,p_2(x)}$ by $\delta^{\mathrm{univ}}_{p_2(x)} : \mathcal{O}_K^{\times} \to \widehat{\mathcal{O}}^{\times}_{\mathcal{W}_E,p_2(x)}$. The E-algebra $\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}$ is topologically generated by $\{\delta^{\mathrm{univ}}_{p_2(x)}(a) - \delta_x(a) | a \in \mathcal{O}_K^{\times}\}$. We take a quotient $\bar{R}$ of $R_{V_x,\mathcal{T}_x} \hat{\otimes}_E \widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)} \hat{\otimes}_E \widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)}$ by the ideal generated by $\delta_{\mathcal{T}_x}(a) \otimes 1 \otimes 1 - 1 \otimes \delta^{\mathrm{univ}}_{p_2(x)}(a) \otimes 1$ (any $a \in \mathcal{O}_K^{\times}$) and $\lambda_{\mathcal{T}_x} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes Y$, where Y is the canonical coordinate of $\mathbb{G}^{an}_{m,E}$. Then, we can see that the composition of $R_{V_x,\mathcal{T}_x} \to R_{V_x,\mathcal{T}_x} \hat{\otimes}_{E(x)} \widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)} \hat{\otimes} \widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},E(x)} : z \mapsto z \otimes 1 \otimes 1$ with the natural quotient map $R_{V_x,\mathcal{T}_x} \hat{\otimes}_{E(x)} \widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)} \hat{\otimes}_{E(x)} \widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)} \to \bar{R}$ is an isomorphism $R_{V_x,\mathcal{T}_x} \xrightarrow{\sim} \bar{R}$ and, if we denote by $\bar{\delta}^{\mathrm{univ}}_{p_2(x)} : \mathcal{O}_K^{\times} \to \bar{R}^{\times}$ and $\bar{Y} \in \bar{R}^{\times}$ the reduction of $1 \otimes \delta^{\mathrm{univ}}_{p_2(x)} \otimes 1$ and $1 \otimes 1 \otimes Y$, then the universal triangulation on $R_{V_x,\mathcal{T}_x}$ is transformed to (we drop the notation $n \in \mathbb{Z}_{\geqq 1}$)

$$0 \subseteq W(\bar{\delta}^{\mathrm{univ}}_{p_2(x)} \delta_{\bar{Y}}) \subseteq W((p_1^* \tilde{V}^{\mathrm{univ}}) \otimes_{\mathcal{O}_{X(\bar{\rho})}} \bar{R}).$$

We put $V_{\bar{R}} := (p_1^* \tilde{V}^{\mathrm{univ}}) \otimes_{\mathcal{O}_{X(\bar{\rho})}} \bar{R}$ and put $\bar{R}_n := \bar{R}/\mathfrak{m}^n$ and $V_{\bar{R}_n} := V_{\bar{R}} \otimes_{\bar{R}} \bar{R}_n$. Under this situation, first we claim that the natural map $\mathrm{Spm}(\bar{R}_n) \to X(\bar{\rho})$ factors through $X_0(\bar{\rho})$ for any $n \geqq 1$. This follows immediately from the facts that $W(V_{\bar{R}_n}(\bar{\bar{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))$ has a triangulation $0 \subseteq W(\bar{\delta}_{\bar{Y}_n}) \subseteq W(V_{\bar{R}_n}(\bar{\bar{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))$ and that $W(\bar{\delta}_{\bar{Y}_n})$ is crystalline with Hodge-Tate weight zero, where $\bar{Y}_n \in \bar{R}_n$ is the reduction of $\bar{Y}$. From this, $D_{\mathrm{cris}}(W(\bar{\delta}_{\bar{Y}_n})) = D_{\mathrm{cris}}(W(\bar{\delta}_{\bar{Y}_n}))^{\varphi_K = \bar{Y}_n} \cap \mathrm{Fil}^0 D_{dR}(W(\bar{\delta}_{\bar{Y}_n}))$ is a φ-stable free rank one $K_0 \otimes_{\mathbb{Q}_p} \bar{R}_n$ sub module of $D_{\mathrm{cris}}(V_{\bar{R}_n}(\bar{\bar{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))^{\varphi_K = \bar{Y}_n}$ which is contained in $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\bar{R}_n}(\bar{\bar{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))$. Hence, by Lemma 3.8, we have a natural

inclusion

$$D_{\mathrm{cris}}(W(\delta_{\bar{Y}_n})) \subseteq D^+_{\mathrm{cris}}(V_{\bar{R}_n}(\tilde{\tilde{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))^{\varphi_K=\bar{Y}_n} \ \cdots (1).$$

Next, we take an affinoid open $\mathrm{Spm}(R) \subseteq X_0(\bar{\rho})$ which contains x and satisfies the condition in the construction of X_{fs} (see the paragraph after the proof of Lemma 3.12). Let J be the ideal of R which determines $\mathrm{Spm}(R)_{fs}$. We claim that the natural map $R \to \bar{R}$ factors through $R/J \to \bar{R}$, which proves that $x \in \mathcal{E}(\bar{\rho})(E)$ because x is the point determined by the map $R \to \bar{R} \to \bar{R}/\mathfrak{m}$. By construction of J, it suffices to show the following lemma. $\square$

Lemma 3.17. *In the above situation, the following hold:*

(i) *For any $k \geqq 1$ and for any $\sigma \in \mathcal{P}$, the natural map*

$$\varprojlim_n (B^+_{\max,K} \otimes_{K,\sigma} V_{\bar{R}_n}(\tilde{\tilde{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))^{\varphi_K=\bar{Y},G_K} \to \varprojlim_n (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} V_{\bar{R}_n}(\tilde{\tilde{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))^{G_K}$$

is surjection.

(ii) *For any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, $Q_\sigma(i)$ is nonzero in $\bar{R}$.*

Proof. Because $\bar{R} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}$ is domain, (i) follows from (ii) and from the above inclusion (1) by the same argument as in the proof of Proposition 2.8 of [Ki03]. We prove (ii). We can write $Q_\sigma(T) = T - \bar{a}_{1,\sigma}$ for the reduction $\bar{a}_{1,\sigma} \in \bar{R}$ of $a_{1,\sigma} \in \mathcal{O}_{X(\bar{\rho})}$. Because, for any $n \in \mathbb{Z}_{\geqq 1}$, $V_{\bar{R}_n}(\tilde{\tilde{\delta}}^{\mathrm{univ}-1}_{p_2(x)})$ has a triangulation $0 \subseteq W(\delta_{\bar{Y}_n}) \subseteq W(V_{\bar{R}_n}(\tilde{\tilde{\delta}}^{\mathrm{univ}-1}_{p_2(x)}))$, $\bar{a}_{1,\sigma} \in \bar{R}_n$ is the σ-part of the Hodge-Tate weight of $\det(V_{\bar{R}_n})(\tilde{\tilde{\delta}}^{\mathrm{univ}-2}_{p_2(x)})$. By Lemma 3.18 below, if we denote by $\delta_0 := (\det(V_x)|_{\mathcal{O}_K^\times})/\delta_x^2 : \mathcal{O}_K^\times \to E^\times$, $\bar{a}_{1,\sigma} \in \bar{R}$ is the image of the σ-part of Hodge-Tate weight $a^{\mathrm{univ}}_\sigma \in R_{\delta_0}$ of the universal deformation $\delta^{\mathrm{univ}}_{\delta_0} : \mathcal{O}_K^\times \to R^\times_{\delta_0}$ by the injection $R_{\delta_0} \hookrightarrow R_{V_x,\mathcal{T}_x} \xrightarrow{\sim} \bar{R}$ induced by a morphism $f : D_{V_x,\mathcal{T}_x} \to D_{\delta_0}$ defined below, where the injection follows from Lemma 3.18 below. Hence, for any $i \in \mathbb{Z}_{\leqq 0}$, $Q_\sigma(i) = (i - \bar{a}_{1,\sigma}) \neq 0 \in \bar{R}$ by Lemma 3.19 below. $\square$

Let $\delta_0 : \mathcal{O}_K^\times \to E^\times$ be a continuous homomorphism. We define a functor $D_{\delta_0} : \mathcal{C}_E \to Sets$ by

$$D_{\delta_0}(A) := \{\delta_A : \mathcal{O}_K^\times \to A^\times : \text{ continuous homomorphisms } \delta_A \pmod{\mathfrak{m}_A} = \delta_0\}.$$

It is easy to show that this functor is pro-representable by a ring R_{δ_0} which is isomorphic to $E[[T_1, T_2, \cdots, T_d]]$ where $d := [K : \mathbb{Q}_p]$. Let W be a rank two split trianguline E-B-pair with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W$ such that $W/W(\delta_1) \xrightarrow{\sim} W(\delta_2)$ for some continuous homomorphisms $\delta_1, \delta_2 : K^\times \to E^\times$. We put $\delta_0 := (\delta_2/\delta_1)|_{\mathcal{O}_K^\times}$. We define a morphism of functors $f : D_{W,\mathcal{T}} \to D_{\delta_0}$ as follows. Let $[(W_A, \psi_A, \mathcal{T}_A)] \in D_{W,\mathcal{T}}(A)$ be an equivalent class of trianguline deformation of $(W, \mathcal{T})$ over A with a triangulation $\mathcal{T}_A : 0 \subseteq W(\delta_{1,A}) \subseteq W_A$ such that $W_A/W(\delta_{1,A}) \xrightarrow{\sim} W(\delta_{2,A})$, then we define f by

$$f([(W_A, \psi_A, \mathcal{T}_A]) := (\delta_{2,A}/\delta_{1,A})|_{\mathcal{O}_K^\times} \in D_{\delta_0}(A).$$

Lemma 3.18. *Let W be a two dimensional split trianguline E-B-pair with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W$ such that $W/W(\delta_1) \xrightarrow{\sim} W(\delta_2)$. If $\mathrm{H}^2(G_K, W(\delta_1/\delta_2)) = 0$, then the morphism of functors $f : D_{W,\mathcal{T}} \to D_{\delta_0}$ defined above is formally smooth.*

Proof. Let $A \in \mathcal{C}_E$ and I be an ideal of A such that $I\mathfrak{m}_A = 0$. If we are given $[(W_{A/I}, \psi_{A/I}, \mathcal{T}_{A/I})] \in D_{W,\mathcal{T}}(A/I)$ such that $\mathcal{T}_{A/I} : 0 \subseteq W(\delta_{1,A/I}) \subseteq W_{A/I}$ and $W_{A/I}/W(\delta_{1,A/I}) \xrightarrow{\sim} W(\delta_{2,A/I})$ and are given $\delta_A \in D_{\delta_0}(A)$ such that $\delta_A \otimes_A A/I = (\delta_{2,A/I}/\delta_{1,A/I})|_{\mathcal{O}_K^\times}$. Because D_{δ_0} is formally smooth for any δ_0, there exists a $\delta_{1,A} : K^\times \to A^\times$ such that $\delta_{1,A} \otimes_A A/I = \delta_{1,A/I}$. We take a lift $\lambda \in A^\times$ of $\delta_{2,A/I}(\pi_K) \in (A/I)^\times$ and define $\delta_{2,A} : K^\times \to A^\times$ by $\delta_{2,A}(\pi_K) = \lambda$ and $\delta_{2,A}|_{\mathcal{O}_K^\times} = \delta_A\delta_{1,A}|_{\mathcal{O}_K^\times}$, then we have a short exact sequence of E-B-pairs

$$0 \to W(\delta_1/\delta_2) \otimes_E I \to W(\delta_{1,A}/\delta_{2,A}) \to W(\delta_{1,A/I}/\delta_{2,A/I}) \to 0.$$

Because $\mathrm{H}^2(G_K, W(\delta_1/\delta_2)) = 0$ by the assumption, we have a surjection $\mathrm{H}^1(G_K, W(\delta_{1,A}/\delta_{2,A})) \to \mathrm{H}^1(G_K, W(\delta_{1,A/I}/\delta_{2,A/I}))$. This surjection means that there exists a $[(W_A, \psi_A, \mathcal{T}_A)] \in D_{W,\mathcal{T}}(A)$ which is a lift of $[(W_{A/I}, \psi_{A/I}, \mathcal{T}_{A/I})] \in D_{W,\mathcal{T}}(A/I)$ satisfying that $f([W_A, \psi_A, \mathcal{T}_A]) = (\delta_{2,A}/\delta_{1,A})|_{\mathcal{O}_K^\times} = \delta_A$. We finish the proof of lemma. □

Let $A \in \mathcal{C}_E$ and $\delta : \mathcal{O}_K^\times \to A^\times$ be a continuous homomorphism, then it is known that this is locally $\mathbb{Q}_p$-analytic by Proposition 8.3 of [Bu07]. Then, for any $\sigma \in \mathcal{P}$, we define the σ-component of Hodge-Tate weight of δ by $\frac{\partial\delta(x)}{\partial\sigma(x)}|_{x=1} \in A$, which is equal to the σ-part of Hodge-Tate weight of $A(\tilde{\delta})$ where $\tilde{\delta} : G_K^{\mathrm{ab}} \to A^\times$ is any characters such that $\tilde{\delta} \circ \mathrm{rec}_K|_{\mathcal{O}_K^\times} = \delta$ by Proposition 3.3 of [Na11].

Lemma 3.19. *Let $\delta_0 : \mathcal{O}_K^\times \to E^\times$ be a continuous homomorphism. Let R_{δ_0} be the universal deformation ring of D_{δ_0}. Let $\delta_0^{\mathrm{univ}} : \mathcal{O}_K^\times \to R_{\delta_0}^\times$ be the universal deformation of δ_0. Then, for any $\sigma \in \mathcal{P}$, the σ-part of Hodge-Tate weight $a_\sigma^{\mathrm{univ}} \in R_{\delta_0}$ of δ_0^{univ} (i.e. the projective limit of those of $\delta_0^{\mathrm{univ}} \otimes_{R_{\delta_0}} R_{\delta_0}/\mathfrak{m}^n$ for all $n \in \mathbb{Z}_{\geqq 1}$) is not contained in E, i.e. not constant.*

Proof. Let $a := \{a_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} E$ be any element, then we define a deformation of δ_0 over $E[\varepsilon]$ by

$$\delta_a : \mathcal{O}_K^\times \to E[\varepsilon]^\times : \delta_a(x) := \delta_0(x)(1 + (\sum_{\sigma\in\mathcal{P}} a_\sigma \log(\sigma(x)))\varepsilon).$$

The σ-part of Hodge-Tate weight of δ_a is $\frac{\partial\delta_0(x)}{\partial\sigma(x)}|_{x=1} + a_\sigma\varepsilon$. The lemma follows from this. □

Corollary 3.20. *Let $x = [V_x] \in \mathfrak{X}(\bar{\rho})$ be a point such that V_x is a crystabelline $E(x)$-trianguline representation satisfying the conditions (1) of Definition 2.44. Then, the point $x_\tau := ([V_x], \delta_{\tau,1}|_{\mathcal{O}_K^\times}, \delta_{\tau,1}(\pi_K)) \in X(\bar{\rho})$ is contained in $\mathcal{E}(\bar{\rho})$ for any $\tau \in \mathfrak{S}_2$, where we denote the triangulation $\mathcal{T}_\tau$ by $0 \subseteq W(\delta_{\tau,1}) \subseteq W(V_x)$.*

Proof. This follows from (2) of the above theorem and from Lemma 2.47. □

Next, we describe the local structure of $\mathcal{E}(\bar{\rho})$ at the points satisfying the conditions (i), (ii) in (2) of Theorem 3.16 by the universal trianguline deformation rings. We prove the following theorem, which is a generalization of Proposition 10.6 of [Ki03].

Theorem 3.21. *Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be a point such that the conditions (i), (ii) in (2) of Theorem 3.16 hold. Then, we have a canonical $E(x)$-algebra isomorphism*

$$\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}.$$

In particular, $\mathcal{E}(\bar{\rho})$ is smooth of dimension $3[K:\mathbb{Q}_p]+1$ at x.

Proof. We may assume that $E = E(x)$. We have already showed that the natural map $\widehat{\mathcal{O}}_{X(\bar{\rho}),x} \to \bar{R} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}$ factors through $\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \to \bar{R} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}$ in the proof of Theorem 3.16.

We prove the existence of the inverse map $R_{V_x,\mathcal{T}_x} \xrightarrow{\sim} \bar{R} \to \hat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}$. Because x is an E-rational point, then we can take a Y-small affinoid neighborhood $\mathrm{Spm}(R)$ of x in $\mathcal{E}(\bar{\rho})$. By Proposition 3.7 and Proposition 3.14, there exists a sufficiently large $k > 0$ such that, for any $\sigma \in \mathcal{P}$, there exists a short exact sequence of Banach R-modules with property (Pr)

$$0 \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to U_{k,\sigma} \to 0$$

and that we have a natural isomorphism

$$K\otimes_{K_0} D^+_{\mathrm{cris}}(V_R(\tilde{\delta}^{\mathrm{univ}-1}))^{\varphi_K=Y} \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p} V_R(\tilde{\delta}^{\mathrm{univ}-1}))^{G_K}$$

and that, for any $\sigma \in \mathcal{P}$, we have an ideal $H_\sigma \subseteq R$ defined in Proposition 3.14 satisfying that $\mathrm{Spm}(R)\setminus V(H)$ ($H := \prod_{\sigma\in\mathcal{P}} H_\sigma$) and $\mathrm{Spm}(R)\setminus V(H_\sigma)$ are scheme theoretically dense in $\mathrm{Spm}(R)$.

Under this situation, we prove the existence of the inverse. First, we claim that $D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1}))^{\varphi_K=\lambda_x}$ is a free $K_0\otimes_{\mathbb{Q}_p}E$-module of rank one. By the definition and by Lemma 3.8, this module has a sub module $D_{\mathrm{cris}}(W(\delta_{\lambda_x})) = D_{\mathrm{cris}}(W(\delta_{\lambda_x}))^{\varphi_K=\lambda_x} \cap \mathrm{Fil}^0 D_{\mathrm{dR}}(W(\delta_{\lambda_x}))$ which is rank one. Hence $D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1}))^{\varphi_K=\lambda_x}$ is rank one or two. If this is rank two, then $V_x(\tilde{\delta}_x^{-1})$ is crystalline with Hodge-Tate weight $\{0,k_\sigma\}_{\sigma\in\mathcal{P}}$ such that $k_\sigma \in \mathbb{Z}_{\leqq 0}$ for any $\sigma\in\mathcal{P}$ with unique relative Frobenius eigenvalue λ_x. But then, this implies that $\delta_2/\delta_1 = \prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$, this contradicts the assumption on $(V_x,\mathcal{T}_x)$. Hence $D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1}))^{\varphi_K=\lambda_x}$ is rank one and the inclusion $D_{\mathrm{cris}}(W(\delta_{\lambda_x})) \hookrightarrow D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1}))^{\varphi_K=\lambda_x}$ is isomorphism.

In the same way as in the proof of Proposition 10.6 of [Ki03], we first take the blow up $\tilde{T}$ of $\mathrm{Spm}(R)$ along H. By the definition of blow up, for any point $\tilde{x}\in\tilde{T}$ above $x\in\mathrm{Spm}(R)$ and for any $\sigma\in\mathcal{P}$, there exists an element $f_\sigma \in H_\sigma$ such that f_σ is a non zero divisor of $\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}$ and $H_\sigma\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}} = f_\sigma\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}$. By the definition of H_σ,

for any $\sigma \in \mathcal{P}$ and for any $\tilde{x} \in \tilde{T}$ above x, there exists a G_K-equivariant map $V_R(\tilde{\delta}^{\mathrm{univ}-1})^{\vee} \to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y}$ such that the composite with

$$\begin{aligned}(B^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} &\to (B^+_{\max,K}\hat{\otimes}_{K,\sigma}f_\sigma\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}})^{\varphi_K=Y}\\ &\xrightarrow{\sim} (B^+_{\max,K}\hat{\otimes}_{K,\sigma}\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}})^{\varphi_K=Y} \to (B^+_{\max,K}\otimes_{K,\sigma}E(\tilde{x}))^{\varphi_K=Y(\tilde{x})}\end{aligned}$$

is not zero, where the isom $(B^+_{\max,K}\hat{\otimes}_{K,\sigma}f_\sigma\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}})^{\varphi_K=Y} \xrightarrow{\sim} (B^+_{\max,K}\hat{\otimes}_{K,\sigma}\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}})^{\varphi_K=Y}$ is given by $a \mapsto \frac{a}{f_\sigma}$. Using this map and the fact that $D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1}))^{\varphi^f=\lambda_x}$ is rank one, by the induction on n that $D^+_{\mathrm{cris}}(V_R(\tilde{\delta}^{\mathrm{univ}-1})\otimes_R\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})^{\varphi^f=Y}$ is a free $K_0\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}}$-module of rank one and the natural base change map

$$D^+_{\mathrm{cris}}(V_R(\tilde{\delta}^{\mathrm{univ}-1})\otimes_R\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})^{\varphi^f=Y}\otimes_{\mathcal{O}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}}}E(\tilde{x}) \xrightarrow{\sim} D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1})\otimes_E E(\tilde{x}))^{\varphi_K=Y(x)}$$

is isomorphism. Because $\mathrm{Fil}^1(K\otimes_{K_0}D^+_{\mathrm{cris}}(V_x(\tilde{\delta}_x^{-1}))^{\varphi_K=Y(x)}) = \mathrm{Fil}^1D_{\mathrm{dR}}(W(\delta_{\lambda_x})) = 0$, then $D^+_{\mathrm{cris}}(V_R(\tilde{\delta}^{\mathrm{univ}-1})\otimes_R\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})^{\varphi^f=Y}$ is a $(\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})$-filtered φ-module of rank one such that $\mathrm{Fil}^0 = K\otimes_{K_0}D^+_{\mathrm{cris}}(V_R(\delta^{\mathrm{univ}-1})\otimes_R\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})^{\varphi^f=Y}$ and $\mathrm{Fil}^1=0$. By Lemma 2.22, this shows that $V_R\otimes_R\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}$ is the projective limit of split trianguline $(\mathcal{O}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})$-representations with triangulations $0\subseteq W(\bar{\delta}^{\mathrm{univ}}_n\delta_{\bar{Y}_n})\subseteq W(V_R\otimes_R\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}/\mathfrak{m}^n_{\tilde{x}})$ (for any $n\in\mathbb{Z}_{\geqq 1}$) which are trianguline deformations of $(V_x,\mathcal{T}_x)\otimes_E E(\tilde{x})$, hence the natural map $R_{V_x}\to\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}\to\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}$ factors through $R_{V_x}\to R_{V_x,\mathcal{T}_x}$ for any $\tilde{x}\in\tilde{T}$ above x. Moreover, because the natural map

$$\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}\hookrightarrow\prod_{\tilde{x}\in\tilde{T},p(\tilde{x})=x}\widehat{\mathcal{O}}_{\tilde{T},\tilde{x}}$$

is injection by Lemma 10.7 of [Ki03] and by (2) of Proposition 3.14 (where $p:\tilde{T}\to\mathrm{Spm}(R)$ is the projection) , the map $R_{V_x}\to\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}$ also factors through $R_{V_x,\mathcal{T}_x}\to\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}$. By this natural construction, we can easily check that this is the inverse of the map giving the above. We finish to prove the existence of the isomorphism $\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}\xrightarrow{\sim}R_{V_x,\mathcal{T}_x}$ for such points. Because this isomorphism is preserved by the base change from E to any finite extension E' by Lemma 3.10, the smoothness around these points follows from this isomorphism and from Lemma 2.8 of [BLR95]. □

4. Zariski density of two dimensional crystalline representations.

In this final section, as an application of Theorem 2.61 (in the two dimensional case) and of Theorem 3.21, we prove Zariski density of two dimensional crystalline representations for any p-adic field.

We define a map $\pi:\mathcal{E}(\bar{\rho})\to\mathcal{W}_E\times_E\mathcal{W}_E$ by $([V_x],\delta_x,\lambda_x)\mapsto(\delta_x,\det(V_x)|_{\mathcal{O}_K^\times}/\delta_x)$.

Proposition 4.1. *For any point $x \in \mathcal{E}(\bar{\rho})$ which satisfies all the conditions of Theorem 3.21, the map $\pi : \mathcal{E}(\bar{\rho}) \to \mathcal{W}_E \times_E \mathcal{W}_E$ is smooth at x.*

Proof. Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be such a point. We put $\delta'_x := \det(V_x)|_{\mathcal{O}_K^\times}/\delta_x$. By the same argument as in Proposition 9.5 of [Ki03], we have an isomorphism $\widehat{\mathcal{O}}_{\mathcal{W}_E \times_E \mathcal{W}_E, (\delta_x, \delta'_x)} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{W}_E, \delta_x} \hat{\otimes}_{E(x)} \widehat{\mathcal{O}}_{\mathcal{W}_E, \delta'_x} \xrightarrow{\sim} R_{\delta_x} \hat{\otimes}_{E(x)} R_{\delta'_x}$. Hence, by Theorem 3.21, the completion of π at x is the morphism

$$\pi : \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}) \to \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{W}_E \times_E \mathcal{W}_E, (\delta_x, \delta'_x)})$$

induced by the morphism of functors

$$\pi_x : D_{V_x, \mathcal{T}_x} \to D_{\delta_x} \times D_{\delta'_x} : [(W_A, \psi_A, \mathcal{T}_A)] \mapsto (\delta_{1,A}|_{\mathcal{O}_K^\times}, \delta_{2,A}|_{\mathcal{O}_K^\times})$$

where $\mathcal{T}_A : 0 \subseteq W(\delta_{1,A}) \subseteq W_A$ and $W_A/W(\delta_{1,A}) \xrightarrow{\sim} W(\delta_{2,A})$ for any $A \in \mathcal{C}_{E(x)}$. Then, we can prove formally smoothness of this morphism of functor in the same way as in Lemma 3.18. Hence, π is smooth at x by Proposition 2.40 and by Proposition 2.9 of [BLR95]. □

Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be an E-rational point such that V_x is a crystalline split trianguline E-representation with a triangulation $\mathcal{T}_x : 0 \subseteq W(\delta_x \delta_{\lambda_x}) \subseteq W(V_x)$ satisfying the conditions (1) of Definition 2.44 (Corollary 3.20). By Proposition 3.14, for any Y-small affinoid open neighborhood $U = \mathrm{Spm}(R)$ of x in $\mathcal{E}(\bar{\rho})$, there exits $k > 0$ and there exists a short exact sequence of Banach R-modules with property (Pr)

$$0 \to K \otimes_{K_0} (B^+_{\max} \hat{\otimes}_{\mathbb{Q}_p} R)^{\varphi_K = Y} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} R \to U_k \to 0$$

and we have a natural isomorphism

$$K \otimes_{K_0} D^+_{\mathrm{cris}}(V_R(\tilde{\delta}_R^{-1}))^{\varphi_K = Y} \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_R(\tilde{\delta}_R^{-1}))^{G_K}$$

and for any $\sigma \in \mathcal{P}$ there exists the smallest ideal $H_\sigma \subseteq R$ satisfying that

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} H_\sigma V_R(\tilde{\delta}_R^{-1}))^{G_K} \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} V_R(\tilde{\delta}_R^{-1}))^{G_K}$$

where we put $V_R := \Gamma(U, p_1^*(\tilde{V}^{\mathrm{univ}}))$ and $\delta_R : \mathcal{O}_K^\times \to R^\times$ is the restriction of $p_2^*(\delta^{\mathrm{univ}})$ to U. Moreover, if we put $Q := \prod_{\sigma \in \mathcal{P}, 0 \leqq i \leqq k} Q_\sigma(-i) \in R$, then we have inclusions $\mathrm{Spm}(R)_Q \subseteq \mathrm{Spm}(R) \setminus V(H_\sigma) \subseteq \mathrm{Spm}(R)$ by the proof of Proposition 3.14. Moreover, shrinking U suitably, we may assume that $v_p(\lambda_y) = v_p(\lambda_x)$ for any $y = (V_y, \delta_y, \lambda_y) \in U$ and that $\pi|_U$ is smooth by Proposition 4.1.

Under this situation, we study the map $\pi|_U : U \to \mathcal{W}_E \times_E \mathcal{W}_E : ([V_y], \delta_y, \lambda_y) \mapsto (\delta_y, \det(V_y)|_{\mathcal{O}_K^\times}/\delta_y)$ around x in detail. Because V_x is crystalline , we can write

$$\pi(x) = (\prod_{\sigma \in \mathcal{P}} \sigma^{k_{1,\sigma}}, \prod_{\sigma \in \mathcal{P}} \sigma^{k_{2,\sigma}}) \in \mathcal{W}_E \times_E \mathcal{W}_E$$

for some integers $\{k_{1,\sigma}, k_{2,\sigma}\}_{\sigma \in \mathcal{P}}$, then we define a subset

$$\begin{aligned}(\mathcal{W}_E \times_E \mathcal{W}_E)_{cl,x} := \{ & (\textstyle\prod_{\sigma \in \mathcal{P}} \sigma^{n_\sigma}, \prod_{\sigma \in \mathcal{P}} \sigma^{n_\sigma - m_\sigma}) \in \mathcal{W}_E \times_E \mathcal{W}_E | n_\sigma \in \mathbb{Z}, \\ & m_\sigma \in \mathbb{Z}_{\geqq k+1} \text{ for any } \sigma \in \mathcal{P} \text{ and } \textstyle\sum_{\sigma \in \mathcal{P}} m_\sigma \geqq 2 e_K v_p(\lambda_x) + [K : \mathbb{Q}_p] + 1 \},\end{aligned}$$

where e_K is the absolute ramified index of K. Then, for any admissible open neighborhood V of $\pi(x)$ in $\mathcal{W}_E \times_E \mathcal{W}_E$, there exists an affinoid open $V' \subseteq V$ which contains $\pi(x)$ such that $V'_{\mathrm{cl},x} := (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x} \cap V'$ is Zariski dense in V'. Under this situation, we prove the following lemma.

Lemma 4.2. *Let $y := (\prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma}, \prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma - m_\sigma})$ be an element in $(\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$ and let $z := ([V_z], \delta_z, \lambda_z)$ be a point in $U \cap \pi^{-1}(y)$, then V_z is crystalline and split trianguline $E(z)$-representation with a triangulation $\mathcal{T}_z : 0 \subseteq W(\delta_z\delta_{\lambda_z}) \subseteq W(V_z)$ which satisfies the conditions* (1) *and* (2) *of Definition 2.44.*

Proof. Let z be such a point. By Corollary 2.6 of [Ki03], we have a natural isomorphism

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_R(\tilde{\delta}_R^{-1}))^{G_K} \otimes_R E(z) \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_z(\tilde{\delta}_z^{-1}))^{G_K}$$

and this is a free $K \otimes_{\mathbb{Q}_p} E(z)$-module of rank one. Because we have an isomorphism

$$K \otimes_{K_0} D^+_{\mathrm{cris}}(V_R(\tilde{\delta}_R^{-1}))^{\varphi_K = Y} \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_R(\tilde{\delta}_R^{-1}))^{G_K}$$

and have an injection

$$K \otimes_{K_0} D^+_{\mathrm{cris}}(V_z(\tilde{\delta}_z^{-1}))^{\varphi_K = \lambda_z} \hookrightarrow (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_z(\tilde{\delta}_z^{-1}))^{G_K}$$

induced from the injection $K \otimes_{K_0} (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E(z))^{\varphi_K = \lambda_z} \hookrightarrow B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E(z)$. From these facts, we obtain an isomorphism

$$K \otimes_{K_0} D^+_{\mathrm{cris}}(V_z(\tilde{\delta}_z^{-1}))^{\varphi_K = \lambda_z} \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_z(\tilde{\delta}_z^{-1}))^{G_K}.$$

On the other hand, because the Hodge-Tate weight of $V_z(\tilde{\delta}_z^{-1})$ is $\{0, -m_\sigma\}_{\sigma\in\mathcal{P}}$ and $m_\sigma \geqq k+1 \geqq 1$, $(t^{k+1} B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_z(\tilde{\delta}_z^{-1}))^{G_K}$ is also a free $K \otimes_{\mathbb{Q}_p} E(z)$-module of rank one. These implies that $D_{\mathrm{dR}}(V_z(\tilde{\delta}_z^{-1}))$ is a free of rank two $K \otimes_{\mathbb{Q}_p} E(z)$-module, i.e. $V_z(\tilde{\delta}_z^{-1})$ is potentially semi-stable and split trianguline with a triangulation $\mathcal{T}'_z : 0 \subseteq W(\delta_{\lambda_z}) \hookrightarrow W(V_z(\tilde{\delta}_z^{-1}))$. Moreover, if we write $\delta_2 := \det(V_z)/\delta_z^2 \delta_{\lambda_z} : K^\times \to E(z)^\times$, then we have $W(V_z(\tilde{\delta}_z^{-1}))/W(\delta_{\lambda_z}) \xrightarrow{\sim} W(\delta_2)$ and $\delta_2|_{\mathcal{O}_K^\times} = \prod_{\sigma\in\mathcal{P}} \sigma^{-m_\sigma}$ because $z \in \pi^{-1}(y)$, hence these facts imply that $V_z(\tilde{\delta}_z^{-1})$ is semi-stable. Finally, we claim that $V_z(\tilde{\delta}_z^{-1})$ is crystalline. If we assume that $V_z(\tilde{\delta}_z^{-1})$ is semi-stable but not crystalline, then φ^f-eigenvalue of $W(\delta_2)$ is $\lambda_z p^f$ or $\lambda_z p^{-f}$. Because of the weakly admissibility of $D_{\mathrm{st}}(V_z(\tilde{\delta}_z^{-1}))$, we have an equality $t_N(V_z(\tilde{\delta}_z^{-1})) = t_H(V_z(\tilde{\delta}_z^{-1}))$. On the other hand, because $W(\det(V_z(\tilde{\delta}_z^{-1}))) \xrightarrow{\sim} W(\delta_{\lambda_z}\delta_2)$, we have $t_N(V_z(\tilde{\delta}_z^{-1})) = \frac{2}{f} v_p(\lambda_z) \pm 1 = \frac{2}{f} v_p(\lambda_x) \pm 1$ and $t_H(V_z(\tilde{\delta}_z^{-1})) = \frac{1}{[K:\mathbb{Q}_p]}(\sum_{\sigma\in\mathcal{P}} m_\sigma)$, hence $t_N(V_z(\tilde{\delta}_z^{-1})) < t_H(V_z(\tilde{\delta}_z^{-1}))$ because $y \in (\mathcal{W} \times_E \mathcal{W}_E)_{\mathrm{cl},x}$, this is a contradiction. Hence, $V_z(\tilde{\delta}_z^{-1})$ is crystalline and, because $\tilde{\delta}_z$ is crystalline, V_z is also crystalline. Finally, twisting $\mathcal{T}'_z$ by δ_z, we obtain a triangulation $\mathcal{T}_z : 0 \subseteq W(\delta_z\delta_{\lambda_z}) \subseteq W(V_z)$ which satisfies (1) and (2) of Definition 2.44. □

Lemma 4.3. *Let $z = ([V_z], \delta_z, \lambda_z)$ be a point in $U \cap \pi^{-1}(y)$ as in the above lemma. Then we have a natural isomorphism $\widehat{\mathcal{O}}_{\pi^{-1}(y),z} \xrightarrow{\sim} R^{\mathrm{cris}}_{V_z}$.*

Proof. First, by Lemma 2.47 and by Theorem 3.21 and by Lemma 4.2, we have a triangulation $\mathcal{T}_z : 0 \subseteq W(\delta_z\delta_{\lambda_z}) \subseteq W(V_z)$ and the functor $D_{V_z,\mathcal{T}_z}$ is represented by $R_{V_z,\mathcal{T}_z}$ and we have an isomorphism $\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),z} \xrightarrow{\sim} R_{V_z,\mathcal{T}_z}$. Then, the completion at z and $\pi(z)$ of the morphism $\pi_z : \mathcal{E}(\bar{\rho}) \to \mathcal{W}_E \times_E \mathcal{W}_E$ is the morphism

$$\pi : \mathrm{Spf}(R_{V_z,\mathcal{T}_z}) \to \mathrm{Spf}(R_{\delta_z}\hat{\otimes}_{E(z)}R_{\delta'_z})$$

induced by

$$D_{V_z,\mathcal{T}_z} \to D_{\delta_z} \times D_{\delta'_z} : [(V_A, \psi_A, \mathcal{T}_A)] \mapsto (\delta_{1,A}|_{\mathcal{O}_K^\times}, \delta_{2,A}|_{\mathcal{O}_K^\times}),$$

where $\delta'_z := \det(V_z)|_{\mathcal{O}_K^\times}/\delta_z$. Under this interpretation, the closed fiber $\mathrm{Spf}(\widehat{\mathcal{O}}_{\pi^{-1}(y),z})$ of π_z corresponds to the sub deformation D' of $D_{V_z,\mathcal{T}_z}$ defined by

$$D'(A) := \{[(V_A, \psi_A, \mathcal{T}_A)] \in D_{V_z,\mathcal{T}_z}(A) | \delta_{1,A}|_{\mathcal{O}_K^\times} = \delta_z \otimes_{E(z)} \mathrm{id}_A,\ \delta_{2,A}|_{\mathcal{O}_K^\times} = \delta'_z \otimes_{E(z)} \mathrm{id}_A\}$$

for any $A \in \mathcal{C}_{E(z)}$. Because V_z is crystalline, this is equivalent to that V_A is crystalline by Lemma 2.56. Therefore we have $D' = D^{\mathrm{cris}}_{V_z}$, hence we have an isomorphism $R^{\mathrm{cris}}_{V_z} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\pi^{-1}(y),z}$. □

In the situation of Lemma 4.3, for any $y := (\prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma}, \prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma - m_\sigma}) \in (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$, we set $U_y := \pi^{-1}(y) \cap U$, which is smooth over $E(y)$ by the assumption on U, and define a subset

$$U_{y,b} := \{z = ([V_z], \delta_z, \lambda_z) \in U_y | V_z \text{ is benign}\}.$$

Proposition 4.4. *In the above situation, if U_y is not empty, then $U_{y,b}$ is an admissible open which is scheme theoretically dense in U_y, in particular $U_{y,b}$ is non-empty.*

Proof. We denote by $U_y := \mathrm{Spm}(R')$. First, by Lemma 4.2, any point $z \in U_y$ satisfies the condition (1) and (2) of Definition 2.44 and V_z is crystalline with Hodge-Tate weight $\{n_\sigma, n_\sigma - m_\sigma\}_{\sigma\in\mathcal{P}}$. Because U_y is smooth, so in particular U_y is reduced. Hence, by Corollary 6.3.3 of [Be-Co08] and by Corollary 3.19 of [Ch09a],

$$D_{\mathrm{cris}}(V_{R'}(\delta_{R'}^{-1})) := \varinjlim_n (\frac{1}{t^n} B^+_{\max}\hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\delta_{R'}^{-1}))^{G_K}$$

is a locally free $K_0 \otimes_{\mathbb{Q}_p} R'$-module of rank two and, for any $z \in U_y$, we have an isomorphism

$$D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1})) \otimes_{R'} E(z) \xrightarrow{\sim} D_{\mathrm{cris}}(V_z(\tilde{\delta}_z^{-1}))$$

and

$$K \otimes_{K_0} D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1})) \xrightarrow{\sim} (B_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K} = (B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K},$$

where the last equality follows from the assumption on the Hodge-Tate weight of V_z for any $z \in U_y$. Because $U_y \subseteq U_Q$, we have an isomorphism

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p} V_R(\tilde{\delta}_R^{-1}))^{G_K} \otimes_R R' \xrightarrow{\sim} (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K},$$

which is a locally free $K\otimes_{\mathbb{Q}_p} R'$-module of rank one by Corollary 2.6 of [Ki03]. Because the natural map $K\otimes_{K_0}(B^+_{\mathrm{max}}\hat{\otimes}_{\mathbb{Q}_p}R')^{\varphi^f=Y}\hookrightarrow B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}R'$ is injection, so we have an isomorphism

$$K\otimes_{K_0} D^+_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y}\xrightarrow{\sim}(B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K}.$$

From these, we can see that the natural map

$$(B^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K}\rightarrow(B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K}$$

is surjection. Hence, we have a short exact sequence

$$0\rightarrow \mathrm{Fil}^kD_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\rightarrow D^+_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\rightarrow(B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K}\rightarrow 0,$$

where we define $\mathrm{Fil}^kD_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1})):=(t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K}$ which is a locally free $K\otimes_{\mathbb{Q}_p}R'$-module of rank one. If we denote by

$$D_2:=D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))/D^+_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y},$$

then the above facts imply that D_2 is also a locally free $K_0\otimes_{\mathbb{Q}_p}R'$-module of rank one. By taking a sufficiently fine affinoid covering of $\mathrm{Spm}(R')$, we may assume that all these modules are free over $K_0\otimes_{\mathbb{Q}_p}R'$ or $K\otimes_{\mathbb{Q}_p}R'$. If we decompose $D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))=\oplus_{\tau:K_0\rightarrow K_0}D_\tau$ etc,then we have a short exact sequence

$$0\rightarrow D_\tau^{+,\varphi^f=Y}\rightarrow D_\tau\rightarrow D_{2,\tau}\rightarrow 0$$

of free R'-modules with an R'-linear φ^f-action for any τ. We take the $Y_1\in R'^{\times}$ such that $\varphi^f(e)=Y_1e$ where $e\in D_{2,\tau}$ is a R'-base of $D_{2,\tau}$. Because Y_1 is a lift of the other Frobenius eigenvalue of $D_{\mathrm{cris}}(V_z(\tilde{\delta}_z^{-1}))$ (one is λ_z) for any $z\in U_y=\mathrm{Spm}(R')$ and because $D_{\mathrm{cris}}(V_z(\tilde{\delta}_z^{-1}))$ is weakly admissible, the condition $\sum_{\sigma\in\mathcal{P}}m_\sigma\geqq 2e_Kv_p(\lambda_z)+[K:\mathbb{Q}_p]+1$ for any $z\in U_y$ implies that

$$Y-Y_1(\text{ and }Y-p^{\pm f}Y_1)\in R'^{\times}.$$

Then, an easy linear algebra implies that there exists a decomposition $D_\tau=R'e'_1\oplus R'e'_2$ such that $R'e'_1=D_\tau^{\varphi^f=Y}=D_\tau^{+,\varphi^f=Y}$ and $R'e'_2=D_\tau^{\varphi^f=Y_1}$. Twisting these by φ^i for any $0\leqq i\leqq f-1$, we obtain a decomposition $D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))=D^+_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y}\oplus D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y_1}$. We denote by e_1 (resp. e_2) a $K_0\otimes_{\mathbb{Q}_p}R'$-base of $D^+_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y}$ (resp. $D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y_1}$). For any $\sigma\in\mathcal{P}$, we denote by $e_{1,\sigma},e_{2,\sigma}$ the R'-basis of the σ-component of $D^+_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\xrightarrow{\sim}K\otimes_{K_0}D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))$ naturally induced from e_1,e_2. Under this situation, we write the σ-component $\mathrm{Fil}^kD_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))_\sigma$ by using the basis $e_{1,\sigma},e_{2,\sigma}$ as follows. Because the natural map $D^+_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))^{\varphi^f=Y}\xrightarrow{\sim}(B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_{R'}(\tilde{\delta}_{R'}^{-1}))^{G_K}$ is isomorphism, the natural map

$$\mathrm{Fil}^kD_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\rightarrow K\otimes_{K_0}D_2,$$

which is the composition of the natural inclusion

$$\mathrm{Fil}^kD_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\rightarrow D_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))=K\otimes_{K_0}D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))$$

with the natural projection $K\otimes_{K_0}D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\rightarrow K\otimes_{K_0}D_2$, is an isomorphism. Hence, for any $\sigma\in\mathcal{P}$, we can take a R'-base of $\mathrm{Fil}^kD_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))_{\sigma}$ of the form $e_{2,\sigma}+a_{\sigma}e_{1,\sigma}$ for some $a_{\sigma}\in R'$. Then, by the definition of benign representations, for any $z\in U_y$, V_z is benign if and only if $\prod_{\sigma\in\mathcal{P}}a_{\sigma}(z)\not=0\in E(z)$ because we have $D_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\otimes_{R'}E(z)\xrightarrow{\sim}D_{\mathrm{dR}}(V_z(\tilde{\delta}_z^{-1}))$ and $D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\otimes_{R'}E(z)\xrightarrow{\sim}D_{\mathrm{cris}}(V_z(\tilde{\delta}_z^{-1}))$ etc. Hence, to finish the proof of this proposition, it is enough to show that $\prod_{\sigma\in\mathcal{P}}a_{\sigma}$ is a non-zero divisor in R'. To prove this claim, it is enough to show that $a_{\sigma}\in\widehat{\mathcal{O}}_{U_y,z}\xrightarrow{\sim}R_{V_z}^{\mathrm{cris}}$ is non-zero for any $\sigma\in\mathcal{P}$, $z\in U_y$ because $R_{V_z}^{\mathrm{cris}}$ is domain. To prove this claim, first we note that we have isomorphisms $D_{\mathrm{cris}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\otimes_{R'}R_{V_z}^{\mathrm{cris}}\xrightarrow{\sim}D_{\mathrm{cris}}(V_{R_{V_z}^{\mathrm{cris}}}(\tilde{\delta}_{R_{V_z}^{\mathrm{cris}}}^{-1})):=\varprojlim_n D_{\mathrm{cris}}(V_{R_{V_z}^{\mathrm{cris}}/\mathfrak{m}^n}(\bar{\tilde{\delta}}_{R_{V_z}^{\mathrm{cris}},n}^{-1}))$ and $D_{\mathrm{dR}}(V_{R'}(\tilde{\delta}_{R'}^{-1}))\otimes_{R'}R_{V_z}^{\mathrm{cris}}\xrightarrow{\sim}D_{\mathrm{dR}}(V_{R_{V_z}^{\mathrm{cris}}}(\tilde{\delta}_{R_{V_z}^{\mathrm{cris}}}^{-1})):=\varprojlim_n D_{\mathrm{dR}}(V_{R_{V_z}^{\mathrm{cris}}/\mathfrak{m}^n}(\bar{\tilde{\delta}}_{R_{V_z}^{\mathrm{cris}},n}^{-1}))$ by construction and by Corollary 6.3.3 of [Be-Co08], where we denote by $\mathfrak{m}$ the maximal ideal of $R_{V_z}^{\mathrm{cris}}$ and denote by $\delta_{R_{V_z}^{\mathrm{cris}}}:\mathcal{O}_K^{\times}\rightarrow(R_{V_z}^{\mathrm{cris}})^{\times}$ the homomorphism induced from $\delta_{R'}$ and denote by $\bar{\delta}_{R_{V_z}^{\mathrm{cris}},n}:\mathcal{O}_K^{\times}\rightarrow(R_{V_z}^{\mathrm{cris}}/\mathfrak{m}^n)^{\times}$ the reduction of $\delta_{R_{V_z}^{\mathrm{cris}}}$. Hence, the claim follows from the following lemma.

□

Lemma 4.5. *Let V be a crystalline E-representation with Hodge-Tate weight $\{0,-k_{\sigma}\}_{\sigma\in\mathcal{P}}$ such that $k_{\sigma}\in\mathbb{Z}_{\geqq 1}$ for any $\sigma\in\mathcal{P}$. If we can write $D_{\mathrm{cris}}(V_{R_V^{\mathrm{cris}}}):=\varprojlim_n D_{\mathrm{cris}}(V_{R_V^{\mathrm{cris}}}/\mathfrak{m}^n)$ by $D_{\mathrm{cris}}(V_{R_V^{\mathrm{cris}}})=K_0\otimes_{\mathbb{Q}_p}R_V^{\mathrm{cris}}e_1\oplus K_0\otimes_{\mathbb{Q}_p}R_V^{\mathrm{cris}}e_2$ such that $\varphi^f(e_1)=\tilde{\lambda}_1e_1,\varphi^f(e_2)=\tilde{\lambda}_2e_2$ for some $\tilde{\lambda}_1,\tilde{\lambda}_2\in(R_V^{\mathrm{cris}})^{\times}$ and that $\mathrm{Fil}^{k_{\sigma}}D_{\mathrm{dR}}(V_{R_V^{\mathrm{cris}}})_{\sigma}$ is generated by $e_{2,\sigma}+a_{\sigma}e_{1,\sigma}$ for $\sigma\in\mathcal{P}$. Then, $a_{\sigma}\not=0\in R_V^{\mathrm{cris}}$ for any $\sigma\in\mathcal{P}$.*

Proof. If we denote by $\lambda_i:=\bar{\tilde{\lambda}}_i\in E^{\times}$ and $\bar{a}_{\sigma}\in E$ the reductions of $\tilde{\lambda}_i$ and a_{σ} by the natural quotient map $R_V^{\mathrm{cris}}\rightarrow E$, then $D_{\mathrm{cris}}(V)=K_0\otimes_{\mathbb{Q}_p}E\bar{e}_1\oplus K_0\otimes_{\mathbb{Q}_p}E\bar{e}_2$ such that $\varphi^f(\bar{e}_i)=\lambda_i\bar{e}_i$ and $\mathrm{Fil}^{k_{\sigma}}D_{\mathrm{dR}}(V)_{\sigma}=E(\bar{e}_{2,\sigma}+\bar{a}_{\sigma}\bar{e}_{1,\sigma})$. For any $b:=\{b_{\sigma}\}_{\sigma\in\mathcal{P}}\in\prod_{\sigma\in\mathcal{P}}E$, we construct a deformation $D(b)$ of $D_{\mathrm{cris}}(V)$ over $E[\varepsilon]$ by $D(b):=D_{\mathrm{cris}}(V)\otimes_E E[\varepsilon]$ as a φ-module and $\mathrm{Fil}^0(K\otimes_{K_0}D(b))=K\otimes_{K_0}D(b)$ and

$$\mathrm{Fil}^1(K\otimes_{K_0}D(b))_{\sigma}=\mathrm{Fil}^{k_{\sigma}}(K\otimes_{K_0}D(b))_{\sigma}:=E[\varepsilon](\bar{e}_{2,\sigma}+(\bar{a}_{\sigma}+b_{\sigma}\varepsilon)\bar{e}_{1,\sigma}),$$

$\mathrm{Fil}^{k_{\sigma}+1}(K\otimes_{K_0}D(b))_{\sigma}=0$. For any b as above, $D(b)$ is a deformation of $D_{\mathrm{cris}}(V)$ over $E[\varepsilon]$. The existence of such deformations implies that $a_{\sigma}\not=0$ for any $\sigma\in\mathcal{P}$.

□

Next, we will prove a proposition concerning Zariski density of benign points in $\mathcal{E}(\bar{\rho})$. Before proving this proposition, we first prove some lemmas concerning general (maybe well-known easy) facts about rigid geometry.

Lemma 4.6. *Let T_n be the n-dimensional closed unit disc defined over E. Then, for any admissible open U of T_n which contains the origin $0:=(0,\cdots,0)\in T_n$, there exists $m>>0$ such that $\{(x_1,\cdots,x_n)\in T_n||x_i|\leqq 1/p^m$ for any $1\leqq i\leqq n\}\subseteq U$.*

Proof. Because U is admissibly covered by rational sub-domains, we may assume that U is itself a rational sub-domain, namely, we may assume that there exist $f_1, \cdots, f_d, g \in E\{\{T_1, \cdots, T_n\}\}$ such that $(f_1, \cdots, f_d, g) = E\{\{T_1, \cdots, T_n\}\}$ and $U = \{x = (x_1, \cdots, x_n) \in T_n || f_i(x)| \leqq |g(x)| \text{ for any } 1 \leqq i \leqq d\}$. Then, the condition $0 \in U$ means that $|f_{i,0}| \leqq |g_0|$ for any i, where $f_{i,0}, g_0 \in E$ are the constant terms of f_i and g. If $g_0 = 0$, then $f_{i,0} = 0$ for any i, this means that $(f_1, \cdots, f_d, g) \subseteq (T_1, T_2 \cdots, T_n)$, which is a contradiction. Hence we have $g_0 \neq 0$ and then, because the norms of coefficients of f_i and g are bounded, there exits $m >> 0$ large enough such that $|f_i(x)| \leqq \max\{|f_{i,0}|, |g_0|\} = |g_0|$ and $|g(x)| = |g_0|$ for any $x = (x_1, \cdots, x_n) \in T_n$ such that $|x_i| \leqq 1/p^m$ for any i, i.e. $\{x \in T_n || x_i| \leqq 1/p^m \text{ for any } i\} \subseteq U$. □

Lemma 4.7. *Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be an E-rational point such that V_x is crystalline trianguline as in before Lemma 4.2 and $U \subseteq \mathcal{E}(\bar{\rho})$ be an admissible open neighborhood of x. Then, there exists an admissible open neighborhood $U' \subseteq U$ of x such that $U'_{\mathrm{cl},x} := U' \cap \pi^{-1}(\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x})$ is Zariski dense in U'.*

Proof. Re-taking smaller U, we may assume that U satisfies the properties as in before Lemma 4.2 and that the morphism $\pi|_U : U \to \mathcal{W}_E \times_E \mathcal{W}_E$ is smooth and U is irreducible smooth of dimension $3[K : \mathbb{Q}_p] + 1$ by Theorem 3.21 and Lemma 4.1. In particular, we may assume that $\pi(U) \subseteq \mathcal{W}_E \times_E \mathcal{W}_E$ is an admissible open by Corollary 5.11 of [BL93]. By definition of $\mathcal{W}_E \times_E \mathcal{W}_E$ and $(\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$ and by Lemma 4.6, if we re-take U smaller, then we may assume that there exists an admissible open neighborhood V of $y := \pi(x)$ which is isomorphic to $T_n \xrightarrow{\sim} V$ where $n := 2[K : \mathbb{Q}_p]$ such that y corresponds to the origin $0 \in T_n$ and that, for any $m \geqq 1$, the set $V_{\mathrm{cl},m} := \{x \in T_n || x_i| \leqq 1/p^m \text{ for any } 1 \leqq i \leqq n\} \cap (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$ is Zariski dense in V and that $\pi(U) \subseteq V$ and that $\pi|_U : U \to V$ factors through an étale morphism $\pi' : U \to V \times_E T_{n'}$, where $n' := [K : \mathbb{Q}_p] + 1$, satisfying $\pi'(x) = (y, 0)$. Because $V_{\mathrm{cl},m}$ is Zariski dense in V for any m, the set $(V \times_E T_{n'})_{\mathrm{cl},m} := \{(y', z) \in V_{\mathrm{cl},m} \times_E T_{n'} || z_i| \leqq 1/p^m \text{ for any } 1 \leqq i \leqq n'\}$ is also Zariski dense in $V \times_E T_{n'}$. Because $\pi'(U)$ is admissible open neighborhood of $(y, 0) \in V \times_E T_{n'}$, there exists $m >> 0$ such that $(V \times_E T_{n'})_{\mathrm{cl},m}$ is contained in $\pi'(U)$ by Lemma 4.6. Then, we have $\pi'^{-1}((V \times_E T_{n'})_{\mathrm{cl},m}) \subseteq \pi^{-1}(V_{\mathrm{cl},m}) \subseteq U_{\mathrm{cl},x}$, then the lemma follows from the following lemma.

□

Lemma 4.8. *Let $f : U := \mathrm{Spm}(B) \to V := \mathrm{Spm}(E\{\{T_1, \cdots, T_n\}\})$ be an étale morphism between E-affinoids for some n. We assume that U is irreducible and reduced. If $V_{\mathrm{cl}} \subseteq V$ is a Zariski dense sub set of V such that $V_{\mathrm{cl}} \subseteq f(U)$, then $f^{-1}(V_{\mathrm{cl}})$ is also Zariski dense in U.*

Proof. By the assumption, the natural map $A := E\{\{T_1, \cdots, T_n\}\} \to \prod_{x \in V_{\mathrm{cl}}} E(x)$ is injection. For proving the lemma, it suffices to show that the kernel of the natural map $B \to \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$ is zero. If I is the kernel of this map, then

the map $A \to B/I \hookrightarrow \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$ is equal to the map $A \hookrightarrow \prod_{x \in V_{\mathrm{cl}}} E(x) \to \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$. Because $V_{\mathrm{cl}} \subseteq f(U)$ by the assumption, the map $\prod_{x \in V_{\mathrm{cl}}} E(x) \to \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$ is injection. Therefore, the map $A \hookrightarrow B/I$ is also injection. Then, we have $\dim(A) \leqq \dim(B/I) (\leqq \dim(B))$ by Lemma 4.9 below. From this, we have $\dim(B/I) = \dim(B)$ because B is étale over A. Because U is irreducible and reduced, we have $I = 0$.

□

Lemma 4.9. *Let $f : Z := \mathrm{Spm}(B') \to \mathrm{Spm}(E\{\{T_1, \cdots, T_n\}\})$ be a morphism of affinoids over E. We assume that the induced map $A := E\{\{T_1, \cdots, T_n\}\} \to B'$ is injection. Then, we have $\dim(A) \leqq \dim(B')$.*

Proof. Because $A \hookrightarrow B'$ is injection, the base change $\mathrm{Frac}(A) \hookrightarrow \mathrm{Frac}(A) \otimes_A B'$ is also injection, in particular, the generic fiber of the morphism of schemes $f_0 : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$ induced from the injection $A \hookrightarrow B'$ is not empty. We denote by x the generic point of $\mathrm{Spec}(A)$ and take a point $y \in f_0^{-1}(x)$. Then, by Proposition 2.1.1 of [Berk93], if we denote by $\kappa(x)$ and $\kappa(y)$ the residue fields (in the sense of scheme) at x and y, then the natural inclusion $\kappa(x) \hookrightarrow \kappa(y)$ is an inclusion of valuation fields which induces an inclusion $\tilde{\kappa}(x) \hookrightarrow \tilde{\kappa}(y)$, where $\tilde{\kappa}(-)$ is the residue field of the valuation field $\kappa(-)$. Form this inclusion, we have $(\dim(A) = n =) s(\tilde{\kappa}(x)/E) \leqq s(\tilde{\kappa}(y)/E)$, where $s(\tilde{\kappa}(-)/E)$ is the transcendence degree of $\tilde{\kappa}(-)$ over E. Then, by Lemma 2.5.2 of [Berk93], we also have $s(\tilde{\kappa}(y)/E) \leqq \dim(B')$, hence we have $\dim(A) \leqq \dim(B')$.

□

We set

$$\mathcal{E}(\bar{\rho})_{\mathrm{b}} := \{x \in \mathcal{E}(\bar{\rho}) | V_x \text{ is benign and crystalline } \}.$$

Proposition 4.10. *Let x be an E-rational point in $\mathcal{E}(\bar{\rho})$ as in Lemma 4.7 and U be an admissible open neighborhood of x. If we take an affinoid neighborhood $U' := \mathrm{Spm}(R)$ of x as in Lemma 4.7. Then, $U'_{\mathrm{b}} := \mathcal{E}(\bar{\rho})_{\mathrm{b}} \cap U'$ is also Zariski dense in U'.*

Proof. Consider any element $f \in R$ in the kernel of the natural map $R \to \prod_{z \in U_{\mathrm{b}}} E(z)$. Then, for any $y \in (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x} \cap \pi(U)$, $f|_{\pi^{-1}(y) \cap U} \in \mathcal{O}_{\pi^{-1}(y) \cap U}$ is equal to zero by Proposition 4.4 because $\mathcal{O}_{\pi^{-1}(y) \cap U}$ is reduced. Hence, we obtain $f = 0 \in R$ by Lemma 4.7. □

Corollary 4.11. *Let Y be the Zariski closure of $\mathcal{E}(\bar{\rho})_{\mathrm{b}}$ in $\mathcal{E}(\bar{\rho})$. Then, Y is a union of irreducible components of $\mathcal{E}(\bar{\rho})$.*

Proof. This follows from Proposition 4.10. □

We denote by

$$\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}} := \begin{array}{l} \{x \in \mathfrak{X}(\bar{\rho}) | V_x \text{ is crystalline and the Hodge-Tate weight} \\ \quad \text{of } V_x \text{ is } \{k_{1,\sigma}, k_{2,\sigma}\}_{\sigma \in \mathcal{P}} \text{ such that } k_{1,\sigma} \neq k_{2,\sigma} \text{ for any } \sigma \in \mathcal{P}\}, \end{array}$$

$$\mathfrak{X}(\bar{\rho})_{\mathrm{b}} := \ \{x \in \mathfrak{X}(\bar{\rho}) | V_x \text{ is benign and crystalline } \}.$$

Lemma 4.12. *If $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is not empty, then $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is also not empty.*

Proof. The proof is similar to that of Proposition 4.4. Let $x \in \mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ be a point. Twisting by a suitable character, we may assume that the Hodge-Tate weight of V_x is $\tau := \{0, -k_\sigma\}_{\sigma \in \mathcal{P}}$ such that $k_\sigma \in \mathbb{Z}_{\geqq 1}$. Then, the subset of $\mathfrak{X}(\bar{\rho})$ consisting of points corresponding crystalline representations with Hodge-Tate weight $\tau := \{0, -k_\sigma\}_{\sigma \in \mathcal{P}}$ forms a Zariski closed rigid analytic subspace $\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}$ of $\mathfrak{X}(\bar{\rho})$ by Corollary 2.7.7 of [Ki08]. By the proof of Theorem 3.3.8 of [Ki08], we have an isomorphism $\hat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}, y} \xrightarrow{\sim} R^{\mathrm{cris}}_{V_y}$ for any point $y \in \mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}$. Then, by Corollary 6.3.3 of [Be-Co08] and by Corollary 3.19 of [Ch09a], there exists an admissible open neighborhood U of y in $\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}$, such that $D_{\mathrm{cris}}(V_R) := ((R \hat{\otimes}_{\mathbb{Q}_p} B_{\mathrm{cris}}) \otimes_R V_R)^{G_K}$ is a rank two finite free $K_0 \otimes_{\mathbb{Q}_p} R$-module and $D_{\mathrm{dR}}(V_R) \xrightarrow{\sim} K \otimes_{K_0} D_{\mathrm{cris}}(V_R)$, where V_R is the restriction to U of the universal deformation of $\bar{\rho}$. For any $\sigma' \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, we denote by $D_{\sigma'}$ the σ'-component of $D_{\mathrm{cris}}(V_R)$. We denote by $T^2 - aT + b := \det_R(T\mathrm{id}_{D_{\sigma'}} - \varphi^f|_{D_{\sigma'}}) \in R[T]$ the characteristic polynomial of relative Frobenius on $D_{\sigma'}$, which does not depend on the choice of σ'. Then, we claim that $a^2 - 4b$ is a non zero divisor of R, i.e. the subset $U_{a^2-4b} \subseteq U$ consisting of points z such that $D_{\mathrm{cris}}(V_z)$ has two distinct relative Frobenius eigenvalues is scheme theoretically dense in U. For proving this claim, it suffices to show that, for any $z \in U$, $a^2 - 4b \neq 0$ in $\hat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}, z} \xrightarrow{\sim} R^{\mathrm{cris}}_{V_z}$ because $R^{\mathrm{cris}}_{V_z}$ is domain. But it is easy to see that $D_{\mathrm{cris}}(V_z)$ can be deformed to $E(z)[\varepsilon]$ such that with two distinct relative Frobenius eigenvalues, hence $a^2 - 4b \neq 0$ in $R^{\mathrm{cris}}_{V_z}$. In the same way, we can show that the subset $U'' \subseteq U$ consisting of points z such that $D_{\mathrm{cris}}(V_z)$ has relative Frobenius eigenvalues α_1, α_2 satisfying $\alpha_1 \neq p^{\pm f}\alpha_2$ is also scheme theoretically dense in U, hence their intersection $U_{a^2-4b} \cap U''$ is also scheme theoretically dense in U. Next, we take an element $z \in U_{a^2-4b} \cap U'' \subseteq U$, then by extending scalar, we may assume that $D_{\mathrm{cris}}(V_z) \xrightarrow{\sim} K_0 \otimes_{\mathbb{Q}_p} Ee_{1,z} \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_{2,z}$ such that $\varphi^f(e_{i,z}) = \alpha_{i,z} e_{i,z}$ for some $\alpha_{i,z} \in E^{\times}$ for $i = 1, 2$ such that $\alpha_{1,z} \neq \alpha_{2,z}, p^{\pm f}\alpha_{2,z}$. Then, because $\mathcal{O}_{\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}, z}$ is Henselian by Theorem 2.1.5 of [Berk93], for any sufficiently small affinoid open neighborhood $U' = \mathrm{Spm}(R')$ of z in $U_{a^2-4b} \cap U''$, we have $D_{\mathrm{cris}}(V_R) \otimes_R R' \xrightarrow{\sim} K_0 \otimes_{\mathbb{Q}_p} R'e_1 \oplus K_0 \otimes_{\mathbb{Q}_p} R'e_2$ such that $K_0 \otimes_{\mathbb{Q}_p} R'e_i$ is φ-stable and $\varphi^f(e_i) = \tilde{\alpha}_i e_i$ for some $\tilde{\alpha}_i \in R'^{\times}$ for $i = 1, 2$ satisfying that $\tilde{\alpha}_1 - \tilde{\alpha}_2, \tilde{\alpha}_1 - p^{\pm f}\tilde{\alpha}_2 \in R'^{\times}$. Then, for sufficiently small U', we have $D_{\mathrm{dR}}(V_R) \otimes_R R' \xrightarrow{\sim} D_{\mathrm{dR}}(V_{R'})$ and $\mathrm{Fil}^1 D_{\mathrm{dR}}(V_{R'}) := ((R' \hat{\otimes}_{\mathbb{Q}_p} tB^+_{\mathrm{dR}}) \otimes_{R'} V_{R'})^{G_K} \subseteq D_{\mathrm{dR}}(V_{R'})$ is a free $K \otimes_{\mathbb{Q}_p} R'$-module of rank one which is compatible with any base change by Proposition 2.5 and Corollary 2.6 of [Ki03] and by the proof of Theorem 5.3.2 of [Be-Co08]. Then, for any $\sigma \in \mathcal{P}$, the R'-base of σ-component of $\mathrm{Fil}^1 D_{\mathrm{dR}}(V_{R'})$ can be written by $a_\sigma e_{1,\sigma} + b_\sigma e_{2,\sigma}$ for some $a_\sigma, b_\sigma \in R'$ where $e_{i,\sigma}$ is the σ-component of $e_i \in D_{\mathrm{cris}}(V_R) \otimes_R R' \subseteq D_{\mathrm{dR}}(V_R) \otimes_R R' \xrightarrow{\sim} D_{\mathrm{dR}}(V_{R'})$. Then, by the same argument as in the proof of Proposition 4.4, the subset of U' consisting of benign

representations is $U'_{(\prod_{\sigma\in\mathcal{P}} a_\sigma b_\sigma)}$. Because $\prod_{\sigma\in\mathcal{P}} a_\sigma b_\sigma$ is non zero divisor of R', which can be proved in the same way as in the proof of Proposition 4.4, $U'_{(\prod_{\sigma\in\mathcal{P}} a_\sigma b_\sigma)} \subseteq U'$ is scheme theoretically dense in U', in particular, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is non-empty.

$\square$

Remark 4.13. In the proof of the above lemma, we prove that $\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris,b}} := \{z \in \mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}} | V_z \text{ is benign }\}$ is scheme theoretically dense in $\mathfrak{X}(\bar{\rho})^{\tau}_{\mathrm{cris}}$ for any $\tau := \{k_{1,\sigma}, k_{2,\sigma}\}_{\sigma\in\mathcal{P}}$ such that $k_{1,\sigma} \neq k_{2,\sigma}$ for any $\sigma \in \mathcal{P}$.

For a rigid analytic space Y over E and for a point $y \in Y$, we denote by

$$t_{Y,x} := \mathrm{Hom}_{E(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2, E(y))$$

the tangent space at y, where $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$. The following theorems are the main theorems of this article.

We denote by $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ the Zariski closure of $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ in $\mathfrak{X}(\bar{\rho})$.

Theorem 4.14. *If $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is non empty, then $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ is non empty and a union of irreducible components of $\mathfrak{X}(\bar{\rho})$.*

Proof. First, by Lemma 4.12, $Z := \overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ is non empty. Because any irreducible components have at most $4[K : \mathbb{Q}_p] + 1$ dimension, so it suffices to show that any irreducible components of Z have $4[K : \mathbb{Q}_p]+1$ dimension. Let Z' be an irreducible component of Z. Because the singular locus $Z'_{\mathrm{sing}} \subseteq Z'$ is a proper Zariski closed set in Z', there exists a benign point $x \in \mathfrak{X}(\bar{\rho})_{\mathrm{b}} \cap Z'$ such that Z' (and $\mathfrak{X}(\bar{\rho})$) is smooth at x. By the definition of benign representation and by Theorem 3.16, there exists the different two points

$$x_1 := ([V_x], \delta_{x_1}, \lambda_{x_1}), x_2 := ([V_x], \delta_{x_2}, \lambda_{x_2}) \in \mathcal{E}(\bar{\rho})$$

such that $p_1(x_i) = x$ and with property (ii) in the Theorem 3.16. We denote by Y'_i an irreducible component of $p_1^{-1}(Z)$ containing x_i for $i = 1, 2$ respectively. By Corollary 4.11, these are also irreducible components of $\mathcal{E}(\bar{\rho})$ and Y'_i is unique by Theorem 3.21. Because the natural morphisms $p_1|_{Y'_i} : Y'_i \to \mathfrak{X}(\bar{\rho})$ factor through Z' for $i = 1, 2$, we obtain maps, for $i = 1, 2$,

$$t_{\mathcal{E}(\bar{\rho}),x_i} = t_{Y'_i,x_i} \to t_{Z',x} \hookrightarrow t_{\mathfrak{X}(\bar{\rho}),x}.$$

Hence, we obtain a map

$$\bigoplus_{i=1,2} t_{\mathcal{E}(\bar{\rho}),x_i} \to t_{Z',x} \hookrightarrow t_{\mathfrak{X}(\bar{\rho}),x}.$$

By Theorem 2.61 and Theorem 3.21, this map is surjective, hence we obtain an equality

$$t_{Z',x} = t_{\mathfrak{X}(\bar{\rho}),x}.$$

Because x is smooth at Z', then Z' has dimension $4[K : \mathbb{Q}_p] + 1$.

$\square$

Concerning the assumption that $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is non empty, in this paper we prove the following (maybe well-known) lemma.

Lemma 4.15. *If $\bar{\rho}\otimes_{\mathbb{F}}\bar{\mathbb{F}} \not\sim \begin{pmatrix}\omega & * \\ 0 & 1\end{pmatrix}\otimes\chi$ and $\bar{\rho}\otimes_{\mathbb{F}}\bar{\mathbb{F}} \not\sim \begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\otimes\chi$ for any character $\chi : G_K \to \bar{\mathbb{F}}^\times$, where ω is the mod p cyclotomic character. Then, $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is non empty.*

Proof. First, we prove the absolutely reducible cases. Extending $\mathbb{F}$, we may assume that $\bar{\rho}$ is reducible. Because any character $\chi : G_K \to \mathbb{F}^\times$ has a crystalline lift, we may assume that $\bar{\rho} = \begin{pmatrix}\chi & * \\ 0 & 1\end{pmatrix}$ for a character $\chi : G_K \to \mathbb{F}^\times$ such that $\chi \neq 1$ and $\chi \neq \omega$. Using twists of a Lubin-Tate character of K by $\sigma \in \mathcal{P}$ and a unramified character, we can take a crystalline lift $\tilde{\chi} : G_K \to \mathcal{O}^\times$ of χ whose Hodge-Tate weight is $\{k_\sigma\}_{\sigma\in\mathcal{P}}$ such that $k_\sigma \geqq 1$ for any $\sigma \in \mathcal{P}$. Under the assumption $\chi \neq 1, \omega$, $\mathrm{H}^1(G_K, \mathcal{O}(\tilde{\chi}))$ is a free $\mathcal{O}$-module of rank $[K : \mathbb{Q}_p]$ and the natural map $\mathrm{H}^1(G_K, \mathcal{O}(\tilde{\chi})) \to \mathrm{H}^1(G_K, \mathbb{F}(\chi))$ is surjection. By the choice of Hodge-Tate weight of $\tilde{\chi}$, we have an equality $\mathrm{H}^1_f(G_K, E(\tilde{\chi})) = \mathrm{H}^1(G_K, E(\tilde{\chi}))$. These imply that any extension in $\mathrm{H}^1(G_K, \mathbb{F}(\chi))$ lifts to a extension in $\mathrm{H}^1(G_K, \mathcal{O}(\tilde{\chi}))$ which is crystalline.

Next, we prove the absolutely irreducible case. In this case, if we denote by K_2 the unramified extension of K such that $[K_2 : K] = 2$ and denote by $\chi_2 : G_{K_2}^{\mathrm{ab}} \to \mathbb{F}^\times$ the mod p reduction of the Lubin-Tate character $\chi_{2,\mathrm{LT}}$ of K_2 associated to a uniformizer π_{K_2} of K_2, then it is known that $\bar{\rho} \xrightarrow{\sim} (\mathrm{Ind}_{G_{K_2}}^{G_K}\chi_2^i) \otimes \chi$ for a character $\chi : G_K \to \mathbb{F}^\times$ and for some $i \in \mathbb{Z}$ such that $i \not\equiv 0 \pmod{p^f + 1}$. Hence, it suffices to show that $\mathrm{Ind}_{G_{K_2}}^{G_K}\chi_2^i$ has a crystalline lift. Because χ_2 is the mod p reduction of $\chi_{2,\mathrm{LT}}$, we can take a lift of χ_2^i of the form $\prod_{\sigma\in\mathcal{P}}\tilde{\sigma}(\chi_{2,\mathrm{LT}})^{k_\sigma}$ such that $k_\sigma \geqq 1$ for all $\sigma \in \mathcal{P}$, where $\tilde{\sigma} : K_2 \hookrightarrow \overline{K}$ is an extension of σ. Then, $\mathrm{Ind}_{G_{K_2}}^{G_K}\chi_2^i$ has a crystalline lift $\mathrm{Ind}_{G_{K_2}}^{G_K}(\prod_{\sigma\in\mathcal{P}}\tilde{\sigma}(\chi_{2,\mathrm{LT}})^{k_\sigma})$ whose Hodge-Tate weight is $\{0, k_\sigma\}_{\sigma\in\mathcal{P}}$.

□

We can obtain the following theorem. We remark that, from § 3.3, we assume $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$ but, even if $\mathrm{End}(\bar{\rho}) \neq \mathbb{F}$, we can prove Theorem 3.16 and Theorem 3.21 and Theorem 4.14 without any additional difficulties if we consider the universal framed deformations instead of usual deformations. But, up to now, the author does not know the proof of the following theorem in the case $\mathrm{End}_{G_K}(\bar{\rho}) \neq \mathbb{F}$.

Theorem 4.16. *We assume the following conditions.*

(0) $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$.
(1) $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is not empty.*
(2) $\bar{\rho} \not\sim \begin{pmatrix}1 & * \\ 0 & \omega\end{pmatrix} \otimes \chi$ *for any* $\chi : G_K \to \mathbb{F}^\times$.

(3) *$[K(\zeta_p):K]\neq 2$ or, for any χ, $\bar{\rho}|_{I_K}\not\sim\begin{pmatrix}\chi_2^i & 0\\ 0 & \chi_2^{ip^f}\end{pmatrix}\otimes\chi$ such that $i=\frac{p^f+1}{2}$, where ζ_p is a primitive p-th root of* 1.

Then, we have an equality $\overline{\mathfrak{X}(\bar{\rho})}_{\mathrm{b}}=\mathfrak{X}(\bar{\rho})$.

Proof. First, we prove in the case $\zeta_p\notin K$, i.e. ω is not trivial. In this case, under the conditions of (2) and (3), we have

$$\mathrm{H}^2(G_K,\mathrm{ad}(\bar{\rho}))\xrightarrow{\sim}\mathrm{H}^0(G_K,\mathrm{ad}(\bar{\rho})(\omega))^{\vee}=\mathrm{Hom}_{G_K}(\bar{\rho},\bar{\rho}\otimes\omega)^{\vee}=0.$$

Hence, the universal deformation ring $R_{\bar{\rho}}$ is formally smooth over $\mathcal{O}$ and $\mathfrak{X}(\bar{\rho})$ is isomorphic to $(4[K:\mathbb{Q}_p]+1)$-dimensional open unit disc, in particular irreducible. Hence, by Theorem 4.14, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is Zariski dense in $\mathfrak{X}(\bar{\rho})$.

Next, we prove in the case $\zeta_p\in K$. In this case $\mathfrak{X}(\bar{\rho})$ is not irreducible. Let $P\subset\mathcal{O}_K^{\times}$ be the sub group of $\mathcal{O}_K^{\times}$ consisting of all the p-th power roots of 1 and let p^n be the order of this sub group and let $\zeta_{p^n}\in\mathcal{O}_K^{\times}$ be a primitive p^n-th root of 1. For any $0\leqq i\leqq p^n-1$, we define a sub functor $D_{\bar{\rho},i}$ of $D_{\bar{\rho}}$ by, for any $A\in\mathcal{C}_{\mathcal{O}}$,

$$D_{\bar{\rho},i}(A):=\{(V_A,\psi_A)\in D_{\bar{\rho}}(A)|\det(V_A)(\mathrm{rec}_K(\zeta_{p^n}))=\iota_A(\zeta_{p^n})^i\},$$

where $\iota_A:\mathcal{O}\to A$ is the morphism which gives an $\mathcal{O}$-algebra structure to A. It is easy to see that the canonical inclusion $D_{\bar{\rho},i}\hookrightarrow D_{\bar{\rho}}$ is relatively representable, i.e. this satisfies the conditions (1) and (2) and (3) in the proof of Proposition 2.36. For any i, let $R_{\bar{\rho},i}$ be the quotient of $R_{\bar{\rho}}$ which represents $D_{\bar{\rho},i}$ and let $\mathfrak{X}(\bar{\rho})_i\subseteq\mathfrak{X}(\bar{\rho})$ be the Zariski closed rigid analytic space associated to $R_{\bar{\rho},i}$, then it is easy to see that, as rigid analytic space, $\mathfrak{X}(\bar{\rho})$ is the disjoint union of $\mathfrak{X}(\bar{\rho})_i$ for $0\leqq i\leqq p^n-1$,

$$\mathfrak{X}(\bar{\rho})=\coprod_{0\leqq i\leqq p^n-1}\mathfrak{X}(\bar{\rho})_i.$$

Moreover, we claim that each $\mathfrak{X}(\bar{\rho})_i$ is isomorphic to $(4[K:\mathbb{Q}_p]+1)$-dimensional open unit disc. For proving this claim, it suffices to show that the functor $D_{\bar{\rho},i}$ is formally smooth.

We prove formally smoothness of $D_{\bar{\rho},i}$ as follows. Let A be an object of $\mathcal{C}_{\mathcal{O}}$ and $I\subseteq A$ be a non zero ideal such that $I\mathfrak{m}_A=0$. Let $(V_{A/I},\psi_{A/I})$ be a deformation of $\bar{\rho}$ in $D_{\bar{\rho},i}(A/I)$, then it suffices to show that $(V_{A/I},\psi_{A/I})$ lifts to $D_{\bar{\rho},i}(A)$. Fixing A/I-base of $V_{A/I}$, we represent $V_{A/I}$ by a continuous homomorphism $\rho_{A/I}:G_K\to\mathrm{GL}_2(A/I)$. Because the obstruction of the liftings of $\det(\bar{\rho})$ comes only from that of $\det(\bar{\rho})|_{\mathrm{rec}_K(P)}$, we can take a continuous character $c_A:G_K^{\mathrm{ab}}\to A^{\times}$ which is a lift of $\det(\rho_{A/I})$ and $c_A(\mathrm{rec}_K(\zeta_{p^n}))=\iota_A(\zeta_{p^n})^i$. We take a continuous lift $\tilde{\rho}_A:G_K\to\mathrm{GL}_2(A)$ of $\rho_{A/I}$ such that $\det(\tilde{\rho}_A(g))=c_A(g)$ for any $g\in G_K$ and then we define a 2-cocycle $f:G_K\times G_K\to I\otimes_{\mathbb{F}}\mathrm{ad}(\bar{\rho})$ by

$$\tilde{\rho}_A(g_1g_2)\tilde{\rho}_A(g_2)^{-1}\tilde{\rho}_A(g_1)^{-1}:=1+f(g_1,g_2)\in 1+I\otimes_A\mathrm{M}_2(A)=1+I\otimes_{\mathbb{F}}\mathrm{ad}(\bar{\rho}).$$

Because $\det(\tilde{\rho}_A)=c_A$ is a homomorphism, $f(g_1,g_2)$ is contained in $I\otimes_{\mathbb{F}}\mathrm{ad}^0(\bar{\rho})$, where we denote by $\mathrm{ad}^0(\bar{\rho}):=\{a\in\mathrm{ad}(\bar{\rho})|\mathrm{trace}(a)=0\}$. Hence, we obtain a class

of 2-cocycle $[f]\in \mathrm{H}^2(G_K,\mathrm{ad}^0(\bar{\rho}))$. But, under the assumption (0) and $\zeta_p\in K$, we have

$$\mathrm{H}^2(G_K,\mathrm{ad}^0(\bar{\rho}))\xrightarrow{\sim}\mathrm{H}^0(G_K,\mathrm{ad}^0(\bar{\rho})(\omega))^{\vee}=\mathrm{H}^0(G_K,\mathrm{ad}^0(\bar{\rho}))^{\vee}=0.$$

Hence, twisting $\tilde{\rho}_A$ by using a suitable continuous one cochain $d: G_K\rightarrow I\otimes_{\mathbb{F}}\mathrm{ad}^0(\bar{\rho})$, we obtain a continuous homomorphism $\rho_A: G_K\rightarrow \mathrm{GL}_2(A)$ such that ρ_A is a lift of $\rho_{A/I}$ and $\det(\rho_A)=c_A$, in particular $(V_{A/I,\psi_{A/I}})$ lifts to $D_{\bar{\rho},i}(A)$, which proves the claim.

By this claim and by Theorem 4.14, it suffices to show that, under the assumption (1), $\mathfrak{X}(\bar{\rho})_i\cap\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is non empty for any i. We prove this claim as follows. First, by the assumption (1), there exists i such that $\mathfrak{X}(\bar{\rho})_i\cap\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is non empty. We take a point $x=[V_x]\in\mathfrak{X}(\bar{\rho})_i\cap\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$.

When $p\not=2$, the twist $V_x(\chi_{\mathrm{LT}}^{j(p^f-1)})$ of V_x for any $j\in\mathbb{Z}$ is contained in $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}\cap\mathfrak{X}(\bar{\rho})_{i_j}$, where we define i_j such that $0\leqq i_j\leqq p^n-1$ and $i_j\equiv i+2j(p^f-1)$ (mod p^n). When $p\not=2$, i_j runs through all $0\leqq i'\leqq p^n-1$, hence $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}\cap\mathfrak{X}(\bar{\rho})_{i'}$ is non empty for any i'. Hence, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is Zariski dense in $\mathfrak{X}(\bar{\rho})$ in this case.

Finally, we consider $p=2$ case. By the same argument as above, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}\cap\mathfrak{X}(\bar{\rho})_{i'}$ is non empty for any $0\leqq i'\leqq 2^n-1$ such that $i'\equiv i$ (mod 2). Hence, it suffices to show that there exists i_1 and i_2 such that $0\leqq i_1,i_2\leqq 2^n-1$ and $i_1\not\equiv i_2$ (mod 2) and $\mathfrak{X}(\bar{\rho})_{i_j}\cap\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is non empty for $j=1,2$. We prove this claim as follows.

If ρ is absolutely reducible, then under the assumption of this Theorem, $\rho\sim\begin{pmatrix}\chi & * \\ 0 & 1\end{pmatrix}\otimes\eta$ for some χ and η such that $\chi\not=1$. We may assume that $\eta=1$. By the proof of Lemma 4.15, there exists a crystalline lift $\tilde{\chi}: G_K\rightarrow\mathcal{O}^{\times}$ such that there exists an $x\in\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}\cap\mathfrak{X}(\bar{\rho})_i$ such that $[V_x]\in\mathrm{H}^1(G_K,E(\tilde{\chi}))$ for some i. There also exists an $x'\in\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}\cap\mathfrak{X}(\bar{\rho})_{i'}$ such that $[V_{x'}]\in\mathrm{H}^1(G_K,E(\tilde{\chi}\chi_{\mathrm{LT}}^{2^f-1}))$ and $i\not\equiv i'$ (mod 2). By the same proof as in Lemma 4.12, we can show that $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}\cap\mathfrak{X}(\bar{\rho})_i$ and $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}\cap\mathfrak{X}(\bar{\rho})_{i'}$ are non empty, which proves the above claim. In absolutely irreducible case, we can prove in the same way. Hence, when $p=2$, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is Zariski dense in $\mathfrak{X}(\bar{\rho})$. We finish the proof of this theorem.

□

5. Appendix : Continuous cohomology of B-pairs

In [Na09], we defined a cohomology $\mathrm{H}^i(G_K,W)$ by using continuous cochains which we review below. On the other hand, Liu [Li08] defined another cohomology which we write by $\mathrm{H}^i_{\mathrm{Liu}}(G_K,W):=\mathrm{H}^i(D(W))$ by using a complex defined from the (φ,Γ)-module $D(W)$ associated to W, see 2.1 of [Li08] for the definition. Moreover, he proved that this cohomology satisfies Euler-Poincarè formula and Tate duality. In this appendix, we prove that $\mathrm{H}^i(G_K,W)$ also satisfies Euler-Poincarè formula and Tate duality and that $\mathrm{H}^i(G_K,W)$ is canonically isomorphic to $\mathrm{H}^i_{\mathrm{Liu}}(G_K,W)$.

We recall the definition of $\mathrm{H}^i(G_K, W)$. Let G be a topological group. For a continuous G-module M, we define the group of i-th continuous cochains by

$$\mathrm{C}^i(G, M) := \{c : G^{\times i} \to M | c \text{ is a continuous map } \}.$$

As usual, we define the boundary map

$$\delta : \mathrm{C}^i(G, M) \to \mathrm{C}^{i+1}(G, M)$$

by

$$\begin{aligned}\delta(c)(g_1, g_2, \cdots, g_{i+1}) := \quad & g_1 c(g_2, \cdots, g_{i+1}) + (-1)^{i+1} c(g_1, g_2, \cdots, g_i) \\ & + \textstyle\sum_{s=1}^{i} (-1)^i c(g_1, \cdots, g_{s-1}, g_s g_{s+1}, g_{s+2}, \cdots, g_{i+1}).\end{aligned}$$

Let $W = (W_e, W^+_{\mathrm{dR}})$ be a B-pair and let $W_{\mathrm{dR}} := W_e \otimes_{B_e} B_{\mathrm{dR}}$. For W, we define a complex $\mathrm{C}^\bullet(G_K, W)$ of $\mathbb{Q}_p$-vector spaces as the mapping cone of the map

$$\mathrm{C}^\bullet(G_K, W_e) \oplus \mathrm{C}^\bullet(G_K, W^+_{\mathrm{dR}}) \to \mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}) : (c_e, c_{\mathrm{dR}}) \mapsto c_e - c_{\mathrm{dR}},$$

i.e. we define by

$$\mathrm{C}^0(G_K, W) := \mathrm{C}^0(G_K, W_e) \oplus \mathrm{C}^0(G_K, W^+_{\mathrm{dR}})$$

and by

$$\mathrm{C}^i(G_K, W) := \mathrm{C}^i(G_K, W_e) \oplus \mathrm{C}^i(G_K, W^+_{\mathrm{dR}}) \oplus \mathrm{C}^{i-1}(G_K, W_{\mathrm{dR}})$$

for $i \geqq 1$ and the differentials δ are defined by $\delta : \mathrm{C}^0(G_K, W_e) \oplus \mathrm{C}^0(G_K, W^+_{\mathrm{dR}}) \to \mathrm{C}^1(G_K, W_e) \oplus \mathrm{C}^1(G_K, W^+_{\mathrm{dR}}) \oplus \mathrm{C}^0(G_K, W_{\mathrm{dR}}) (c_e, c_{\mathrm{dR}}) \mapsto (\delta(c_e), \delta(c_{\mathrm{dR}}), c_e - c_{\mathrm{dR}})$ and $\delta : \mathrm{C}^i(G_K, W_e) \oplus \mathrm{C}^i(G_K, W^+_{\mathrm{dR}}) \oplus \mathrm{C}^{i-1}(G_K, W_{\mathrm{dR}}) \to \mathrm{C}^{i+1}(G_K, W_e) \oplus \mathrm{C}^{i+1}(G_K, W^+_{\mathrm{dR}}) \oplus \mathrm{C}^i(G_K, W_{\mathrm{dR}}) (c_e, c_{\mathrm{dR}}, c) \mapsto (\delta(c_e), \delta(c_{\mathrm{dR}}), c_e - c_{\mathrm{dR}} - \delta(c))$ for $i \geqq 1$. We define the cohomology of W by

$$\mathrm{H}^i(G_K, W) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W)),$$

and

$$\mathrm{H}^i(G_K, W_e) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W_e))$$

and

$$\mathrm{H}^i(G_K, W^+_{\mathrm{dR}}) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W^+_{\mathrm{dR}})),\ \mathrm{H}^i(G_K, W_{\mathrm{dR}}) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W_{\mathrm{dR}})).$$

By these definitions, we have the following long exact sequence,

$$\cdots \to \mathrm{H}^{i-1}(K, W_{\mathrm{dR}}) \to \mathrm{H}^i(G_K, W) \to \mathrm{H}^i(G_K, W_e) \oplus \mathrm{H}^i(G_K, W^+_{\mathrm{dR}}) \to \cdots.$$

Before proving Euler-Poincarè formula, we recall some results of [Be09]. Let W be an almost $\mathbb{C}_p$-representation and V_1 and V_2 be $\mathbb{Q}_p$-representations of G_K of dimension d_1 and d_2 respectively and $d \geqq 0$ be an integer such that $V_1 \subseteq W$ and $V_2 \subseteq \mathbb{C}_p^{\oplus d}$ and $W/V_1 \xrightarrow{\sim} \mathbb{C}_p^{\oplus d}/V_2$, then we define the dimension of W by $\dim_{\mathcal{C}(G_K)}(W) := d$ and the height of W by $\mathrm{ht}(W) := d_1 - d_2$. Let $W := (W_e, W^+_{\mathrm{dR}})$ be a B-pair. We define $X_0(W) := W_e \cap W^+_{\mathrm{dR}}$ and $X_1(W) := W_{\mathrm{dR}}/(W_e + W^+_{\mathrm{dR}})$. In [Be09], Berger proved the following theorem.

Theorem 5.1. *Let W be a B-pair of rank d, then :*

(1) $X_0(W)$ *and* $X_1(W)$ *are almost* $\mathbb{C}_p$*-representations,*
(2) *If* W *is pure of slope* $s \leqq 0$*, then* $\dim_{\mathcal{C}(G_K)}(X_0(W)) = -sd$ *and* $\mathrm{ht}(X_0(W)) = d$ *and* $X_1(W) = 0$*,*
(3) *If* W *is pure of slope* $s > 0$*, then* $X_0(W) = 0$ *and* $\dim_{\mathcal{C}(G_K)}(X_1(W)) = sd$ *and* $\mathrm{ht}(X_1(W)) = -d$*.*

Proof. See Theorem 3.1 of [Be09]. $\square$

Lemma 5.2. *Let* W *be an almost* $\mathbb{C}_p$*-representation, then* $\mathrm{H}^i(G_K W)$ *is finite dimensional over* $\mathbb{Q}_p$ *for* $i = 0, 1, 2$ *and zero for* $i \geqq 3$*.*

Proof. This follows from the definition of almost $\mathbb{C}_p$-representations and the facts that $\mathrm{H}^i(G_K, V) = 0$ for $i \geqq 3$ for any $\mathbb{Q}_p$-representation V of G_K and that $\mathrm{H}^i(G_K, \mathbb{C}_p) = 0$ for $i \geqq 2$ and that $\mathrm{H}^i(G_K, V)$ and $\mathrm{H}^i(G_K, \mathbb{C}_p)$ are finite dimensional over $\mathbb{Q}_p$. $\square$

For an almost $\mathbb{C}_p$-representation W, we write $\chi(W) := \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{Q}_p} \mathrm{H}^i(G_K, W)$.

Lemma 5.3. $\chi(W) = -[K : \mathbb{Q}_p]\mathrm{ht}(W)$*.*

Proof. This follows from the definition of almost $\mathbb{C}_p$-representations and Euler-Poincarè formula for $\mathbb{Q}_p$-representations of G_K and the fact that $\chi(\mathbb{C}_p) = 0$. $\square$

Lemma 5.4. *The following equalities hold.*
(1) $\mathrm{C}^\bullet(W_e) = \varinjlim_n \mathrm{C}^\bullet(W_e \cap \frac{1}{t^n} W_{\mathrm{dR}}^+)$.
(2) $\mathrm{C}^\bullet(W_{\mathrm{dR}}) = \varinjlim_n \mathrm{C}^\bullet(G_K, \frac{1}{t^n} W_{\mathrm{dR}}^+)$.

Proof. For any n, $\frac{1}{t^n} W_{\mathrm{dR}}^+$ is closed in $\frac{1}{t^{n+1}} W_{\mathrm{dR}}^+$ and the topology on $\frac{1}{t^n} W_{\mathrm{dR}}^+$ is the topology induced from $\frac{1}{t^{n+1}} W_{\mathrm{dR}}^+$. Hence, by Proposition 5.6 of [Schn01], we obtain the equality (2). For W_e, if we fix an isomorphism $W_e \xrightarrow{\sim} B_e^{\oplus d}$ as B_e-module, the topology on W_e is defined by the direct sum topology of B_e. Because we have an equality $tB_{\max}^{+,\varphi=p^n} = \cap_{m \geqq 0} \mathrm{Ker}(\theta(\varphi^m) : B_{\max}^{+,\varphi=p^{n+1}} \to \mathbb{C}_p)$ by Proposition 8.10 (2) of [Co02], $\frac{1}{t^n} B_{\max}^{+,\varphi=p^n}$ is closed in $\frac{1}{t^{n+1}} B_{\max}^{+,\varphi=p^{n+1}}$ and the topology on $\frac{1}{t^n} B_{\max}^{+,\varphi=p^n}$ is the topology induced from $\frac{1}{t^{n+1}} B_{\max}^{+,\varphi=p^{n+1}}$. Hence, by Proposition 5.6 of [Schn01], we have $\mathrm{C}^\bullet(G_K, W_e) = \varinjlim_n \mathrm{C}^\bullet(G_K, (\frac{1}{t^n} B_{\max}^{+,\varphi=p^n})^{\oplus d}) = \varinjlim_n \mathrm{C}^\bullet(G_K, W_e \cap \frac{1}{t^n} W_{\mathrm{dR}}^+)$. $\square$

Lemma 5.5. *Let* W_{dR}^+ *be a finite free* B_{dR}^+*-module with a continuous semi-linear* G_K*-action. Then the canonical map* $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) \to \varprojlim_n \mathrm{H}^i(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+)$ *is isomorphism.*

Proof. Because we have $\mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}^+) \xrightarrow{\sim} \varprojlim_n \mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+)$, for any $i \geqq 0$, we have the following short exact sequence,

$$0 \to \mathbb{R}^1 \varprojlim_n \mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) \to \varprojlim_n \mathrm{H}^i(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+)) \to 0.$$

Because $\mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+)$ is finite dimensional over $\mathbb{Q}_p$, Mittag-Leffler condition implies that $\mathbb{R}^1 \varprojlim_n \mathrm{H}^{i-1}(K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+) = 0$. The lemma follows from this. $\square$

Corollary 5.6. *Let W_{dR}^+ be as above. Let $\{h_1, h_2, \cdots, h_d\}$ be the generalized Hodge-Tate weight of $W_{\mathrm{dR}}^+/tW_{\mathrm{dR}}^+$. Let $k \geqq 1$ be any integer such that $k + h_j \geqq 0$ for any $h_j \in \mathbb{Z}$. Then the natural map $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}}^+/t^k W_{\mathrm{dR}}^+)$ is isomorphism and $\mathrm{H}^i(G_K, t^{k+1}W_{\mathrm{dR}}^+) = 0$.*

Proof. By the assumption on k, we have $\mathrm{H}^i(G_K, t^l W_{\mathrm{dR}}^+/t^{l+1}W_{\mathrm{dR}}^+) = 0$ for any $l \geqq k+1$. The corollary follows from this and Lemma 5.5. □

Corollary 5.7. *Let W_{dR}^+ be as above, then $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) = \mathrm{H}^i(G_K, W_{\mathrm{dR}}) = 0$ for $i \geqq 2$ and $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+)$ and $\mathrm{H}^i(G_K, W_{\mathrm{dR}})$ are finite dimensional over $\mathbb{Q}_p$ for $i = 0, 1$.*

Proof. Because $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+) = 0$ for $i \geqq 2$ and $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+/t^n W_{\mathrm{dR}}^+)$ is finite dimensional for $i = 0, 1$, we obtain the corollary for W_{dR}^+ by Lemma 5.6. We prove the corollary for W_{dR}. Because $\mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}) \overset{\sim}{\to} \varinjlim_n \mathrm{C}^\bullet(G_K, \frac{1}{t^n}W_{\mathrm{dR}}^+)$ by Lemma 5.4 (2), we have an isomorphism $\mathrm{H}^i(G_K, W_{\mathrm{dR}}) \overset{\sim}{\to} \varinjlim_n \mathrm{H}^i(G_K, \frac{1}{t^n}W_{\mathrm{dR}}^+)$. Then we can show that for n large enough the natural map $\mathrm{H}^i(G_K, \frac{1}{t^{n+j}}W_{\mathrm{dR}}^+) \to \mathrm{H}^i(G_K, \frac{1}{t^{n+j+1}}W_{\mathrm{dR}}^+)$ is isomorphism for any $j \geqq 0$, then the natural map $\mathrm{H}^i(G_K, \frac{1}{t^n}W_{\mathrm{dR}}^+) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}})$ is isomorphism, the corollary for W_{dR} follows from this. □

Lemma 5.8. *Let $W = (W_e, W_{\mathrm{dR}}^+)$ be a B-pair. Then we have $\mathrm{H}^i(G_K, W_e) = 0$ for $i \geqq 3$.*

Proof. Because we have $\mathrm{C}^\bullet(G_K, W_e) = \varinjlim_n \mathrm{C}^\bullet(G_K, W_e \cap \frac{1}{t^n}W_{\mathrm{dR}}^+)$ by Lemma 5.4 (1), we have an isomorphism $\mathrm{H}^i(G_K, W_e) \overset{\sim}{\to} \varinjlim_n \mathrm{H}^i(G_K, W_e \cap \frac{1}{t^n}W_{\mathrm{dR}}^+)$. For any n, because $W_e \cap \frac{1}{t^n}W_{\mathrm{dR}}^+$ is an almost $\mathbb{C}_p$-representation by Theorem 5.1, we have $\mathrm{H}^i(G_K, W_e \cap \frac{1}{t^n}W_{\mathrm{dR}}^+) = 0$ for $i \geqq 3$ by Lemma 5.2. The lemma follows from these facts. □

Theorem 5.9. *Let W be a B-pair, then the following hold.*

(1) *$\mathrm{H}^i(G_K, W)$ is zero for $i \geqq 3$ and $\mathrm{H}^i(G_K, W)$ is finite dimensional over $\mathbb{Q}_p$ for $i = 0, 1, 2$.*
(2) *(Euler-Poincaré characteristic formula)*

$$\sum_{i=0}^{2} \dim_{\mathbb{Q}_p}(-1)^i \mathrm{H}^i(G_K, W) = -[K : \mathbb{Q}_p]\mathrm{rank}(W).$$

Proof. We first prove that $\mathrm{H}^i(G_K, W) = 0$ for $i \geqq 3$. Because there is an exact sequence

$$\cdots \to \mathrm{H}^{i-1}(K, W_{dR}) \to \mathrm{H}^i(G_K, W) \to \mathrm{H}^i(G_K, W_e) \oplus \mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) \to \cdots,$$

the claim follows from Corollary 5.7 and Lemma 5.8. Next we prove that $\mathrm{H}^i(G_K, W)$ is finite dimensional over $\mathbb{Q}_p$. By slope filtration theorem, it suffices to show this

claim when W is pure. Let W be a B-pair pure of slope s. When $s \leqq 0$ we have the following short exact sequence,

$$0 \to W_e \cap W_{\mathrm{dR}}^+ \to W_e \oplus W_{\mathrm{dR}}^+ \to W_{\mathrm{dR}} \to 0$$

by Theorem 5.1 (2). Hence the natural map $\mathrm{H}^i(G_K, W_e \cap W_{\mathrm{dR}}^+) \to \mathrm{H}^i(G_K, W)$ is isomorphism. Because $W_e \cap W_{\mathrm{dR}}^+$ is an almost $\mathbb{C}_p$-representation by Theorem 5.1, $\mathrm{H}^i(G_K, W_e \cap W_{\mathrm{dR}}^+)$ is finite dimensional by Lemma 5.2, this prove the claim for $s \leqq 0$. When $s > 0$, then we have the following short exact sequence

$$0 \to W_e \oplus W_{\mathrm{dR}}^+ \to W_{\mathrm{dR}} \to W_{\mathrm{dR}}/(W_e + W_{\mathrm{dR}}^+) \to 0$$

by Theorem 5.1 (3). Hence we obtain a natural isomorphism $\mathrm{H}^i(K, W) \xrightarrow{\sim} \mathrm{H}^{i-1}(K, W_{\mathrm{dR}}/(W_e + W_{\mathrm{dR}}^+))$. Because $W_{\mathrm{dR}}/(W_e + W_{\mathrm{dR}}^+)$ is an almost $\mathbb{C}_p$-representation by Theorem 5.1, $\mathrm{H}^{i-1}(K, W_{\mathrm{dR}}/(W_e + W_{\mathrm{dR}}^+))$ is finite dimensional, the claim for $s > 0$ follows from this.

Next we prove (2). For W a B-pair or an almost $\mathbb{C}_p$-representation, we write $\chi(W) := \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{Q}_p} \mathrm{H}^i(G_K, W)$. It suffices to show (2) when W is pure of slope s. When $s \leqq 0$, then by the above proof $\chi(W) = \chi(X_0(W))$. By Lemma 5.3 and by Theorem 5.1 (2), $\chi(X^0(W)) = -[K : \mathbb{Q}_p]\mathrm{ht}(W) = -[K : \mathbb{Q}_p]\mathrm{rank}(W)$. When $s > 0$, then by the above proof we have $\chi(W) = -\chi(X^1(W))$. By Lemma 5.3 and by Theorem 5.1 (3), we have $\chi(X^1(W)) = -[K; \mathbb{Q}_p]\mathrm{ht}(X^1(W)) = [K : \mathbb{Q}_p]\mathrm{rank}(W)$. (2) follows from these.

□

Next, we define the cup product pairing for B-pairs $W := (W_e, W_{\mathrm{dR}}^+)$ and $W' := (W'_e, W'^+_{\mathrm{dR}})$ as follows. First, for two continuous cochains $c \in \mathrm{C}^i(G_K, W_?)$ and $c' \in \mathrm{C}^j(G_K, W'_?)$ where $W_?$ is W_e or W_{dR}^+ or W_{dR}, we define a continuous cochain

$$c \cup c' \in \mathrm{C}^{i+j}(G_K, W_? \otimes_{B_?} W'_?)$$

by

$$c \cup c'(g_1, \cdots, g_{i+j}) := c(g_1, \cdots, g_i) \otimes g_1 g_2 \cdots g_i c'(g_{i+1}, \cdots, g_{i+j})$$

where $B_?$ is B_e or B_{dR}^+ or B_{dR} when $W_?$ is W_e or W_{dR}^+ or W_{dR} respectively. Then, $c \cup c'$ satisfies

$$\delta(c \cup c') = \delta(c) \cup c' + (-1)^i c \cup \delta(c').$$

For $c = (c_e, c_{\mathrm{dR}}^+, c_{\mathrm{dR}}) \in \mathrm{C}^i(G_K, W_e) \oplus \mathrm{C}^i(G_K, W_{\mathrm{dR}}^+) \oplus \mathrm{C}^{i-1}(G_K, W_{\mathrm{dR}})$ and $c' = (c'_e, c'^+_{\mathrm{dR}}, c'_{\mathrm{dR}}) \in \mathrm{C}^j(G_K, W'_e) \oplus \mathrm{C}^j(G_K, W'^+_{\mathrm{dR}}) \oplus \mathrm{C}^{j-1}(G_K, W'_{\mathrm{dR}})$ and for a parameter $\gamma \in \mathbb{Q}_p$, we define

$$c \cup_\gamma c' \in \mathrm{C}^{i+j}(G_K, W_e \otimes_{B_e} W'_e) \oplus \mathrm{C}^{i+j}(G_K, W_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+} W'^+_{\mathrm{dR}}) \oplus \mathrm{C}^{i+j-1}(G_K, W_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}} W'_{\mathrm{dR}})$$

by

$$c \cup_\gamma c' := (c_e \cup c'_e, c_{\mathrm{dR}}^+ \cup c'^+_{\mathrm{dR}}, c_{\mathrm{dR}} \cup (\gamma c'_e + (1-\gamma) c'^+_{\mathrm{dR}}) + (-1)^i ((1-\gamma) c_e + \gamma c_{\mathrm{dR}}^+) \cup c'_{\mathrm{dR}})).$$

Then, we can check that if $\delta(c) = \delta(c') = 0$ then $\delta(c \cup_\gamma c') = 0$ and if $\delta c = 0$ and $c' = \delta(c'')$ (or $c = \delta(c'')$ and $\delta(c') = 0$) then $c \cup_\gamma c' \in \mathrm{Im}(\delta)$. Therefore, this paring induces a $\mathbb{Q}_p$-bi-linear paring

$$\cup_\gamma : \mathrm{H}^i(G_K, W) \times \mathrm{H}^j(G_K, W') \to \mathrm{H}^{i+j}(G_K, W \otimes W').$$

Moreover, we can check that $\cup_\gamma$ doesn't depend on the choice of a parameter γ, so we just write $\cup$ instead of $\cup_\gamma$.

We define the paring

$$\cup : \mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \to \mathbb{Q}_p$$

by composing

$$\cup : \mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \to \mathrm{H}^2(G_K, W \otimes W^\vee(\chi_p))$$

with the map $\mathrm{H}^2(G_K, W \otimes W^\vee(\chi_p)) \to \mathrm{H}^2(G_K, W(\mathbb{Q}_p)(\chi_p))$ which is induced from the evaluation map $W \otimes W^\vee(\chi_p) \to W(\mathbb{Q}_p)(\chi_p)$ and with the natural isomorphism $\mathrm{H}^2(G_K, W(\mathbb{Q}_p)(\chi_p)) \xrightarrow{\sim} \mathrm{H}^2(G_K, \mathbb{Q}_p(\chi_p))$ and with Tate's trace map $\mathrm{H}^2(G_K, \mathbb{Q}_p(\chi_p)) \xrightarrow{\sim} \mathbb{Q}_p$ where $W(\mathbb{Q}_p)$ is the B-pair associated to the trivial representation $\mathbb{Q}_p$. Tate duality theorem for B-pairs is following.

Theorem 5.10. *For $i = 0, 1, 2$,the above paring*

$$\cup : \mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \to \mathbb{Q}_p$$

is a perfect paring.

Proof. We can prove this theorem in the same way as in the proof of Theorem 4.7 of [Li08] if we use the Euler-Poincarè formula and the facts that $\mathrm{H}^0(G_K, W(\prod_{\sigma\in\mathcal{P}} \sigma))$ $= 0$ and $\mathrm{H}^0(G_K, W(|\prod_{\sigma\in\mathcal{P}} \sigma|)) = 0$ which have been already proved in Proposition 2.9. □

Finally, we prove that our continuous cohomology is canonically isomorphic to Liu's cohomology. We first define an isomorphism between H^0 by $\mathrm{H}^0_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{Hom}_{\varphi,\Gamma}(R, D(W)) \xrightarrow{\sim} \mathrm{Hom}(W(\mathbb{Q}_p), W) \xrightarrow{\sim} \mathrm{H}^0(G_K, W)$ where $D(W)$ is the (φ, Γ)-module associated to D and R is the trivial (φ, Γ)-module, where the second isomorphism follows from the equivalence of categories between B-pairs and (φ, Γ)-modules.

Theorem 5.11. *The above isomorphism $\mathrm{H}^0_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{H}^0(G_K, W)$ extends uniquely to an isomorphism of δ-functors $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{H}^i(G_K, W)$.*

Proof. This follows from weakly effaceabilities of functors $\mathrm{H}^i_{\mathrm{Liu}}(G_K, -)$ and $\mathrm{H}^i(G_K, -)$. For $\mathrm{H}^i_{\mathrm{Liu}}$, these facts are proved in the proof of Theorem 8.1 of [Ke09]. For $\mathrm{H}^i(G_K, -)$, we can also prove in the same way as in Theorem 8.1 of [Ke09] because we have already proved Euler-Poincarè formula and Tate duality for $\mathrm{H}^i(G_K, -)$ and we have an isomorphism $\mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(W(\mathbb{Q}_p), W)$ which is proved in Proposition 2.2 of [Na09].

□

References

[Bel-Ch09] J.Bellaïche, G.Chenevier, Families of Galois representations and Selmer groups , Astérisque 324 (2009).

[Be02] L.Berger, Représentations p-adiques et équations différentielles, Invent. Math. 148 (2002), 219-284.

[Be08] L.Berger, Construction de (φ, Γ)-modules: représentations p-adiques et B-paires, Algebra and Number Theory, 2 (2008), no. 1, 91–120.

[Be09] L.Berger, Presque $\mathbb{C}_p$-représentations et (φ, Γ)-modules, Journal de l'Institut de Mathématiques de Jussieu 8 (2009), no. 4, 653–668.

[Be11] L.Berger, Trianguline representations, preprint.

[Be-Co08] L.Berger, P.Colmez, Familles de représentations de de Rham et monodromie p-adique, Astérisque 319 (2008), 187-212.

[Berk93] V.Berkovich, Étale cohomology for nonarchimedian analytic spaces, Publications mathématiques de l'IHES 78 (1993).

[BL93] S.Bosch, W.Lütkebohmert, Formal and rigid geometry II. Flattening techniques, Math. Ann. 296, 403-429 (1993).

[BLR95] S.Bosch, W.Lütkebohmert, M.Raynaud, Formal and rigid geometry III. The relative maximum principle, Math. Ann. 302, 1-29 (1995).

[Bu07] K. Buzzard, Eigenvarieties, proceedings of the London Math. Soc. Symp. on L-functions and Galois Representations, Durham (2007).

[Ch09a] G.Chenevier, Une application des variétés de Hecke des groupe unitaires, preprint.

[Ch09b] G.Chenevier, On the infinite fern of Galois representations of unitary type, preprint, arXiv:0911.5726.

[Ch10] G.Chenevier, Sur la densité des représentations cristallines du groupe de Galois absolu de $\mathbb{Q}_p$, preprint, arXiv:1012.2852.

[Co02] P.Colmez, Espaces de Banach de dimension finie, J. Inst. Math. Jussieu 1 (2002), 331-439.

[Co08] P.Colmez, Représentations triangulines de dimension 2, Astérisque 319 (2008), 213-258.

[Co10] P.Colmez, Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules, Astérisque 330 (2010), 281-509.

[Fo94] J.-M. Fontaine, Le corps des périodes p-adiques, Astérisque 223 (1994), 59-111.

[Fo03] J.-M. Fontaine, Presque $\mathbb{C}_p$-représentations. Kazuya Kato's fifties birthday. Doc. Math. 2003, Extra Vol., 285-385 (electronic).

[Ke04] K.Kedlaya, A p-adic local monodromy theorem, Ann. of Math. (2) 160 (2004), 93-184.

[Ke05] K.Kedlaya, Slope filtrations revisited, Doc. Math. 10 (2005), p.447-525 (electronic).

[Ke08] K.Kedlaya, Slope filtrations for relative Frobenius, Astérisque 319 (2008), 259-301.

[Ke09] K.Kedlaya, Some new directions in p-adic Hodge theory, Journal de Thèorie des Nombres de Bordeaux 21 (2009), 285-300.

[Ki03] M.Kisin, Overconvergent modular forms and the Fontaine-Mazur conjecture, Invent. Math. 153 (2003), 373-454.

[Ki08] M.Kisin, Potentially semi-stable deformation rings, J.AMS, 21 (2) (2008), 513-546.

[Ki09] M.Kisin, Moduli of finite flat group schemes and modularity, Ann. of Math. 170 (3) (2009), 1085-1180.

[Ki10] M.Kisin, Deformations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ representations, appendix to [Co10].

[Li08] R.Liu, Cohomology and duality for (φ, Γ)-modules over the Robba ring, Int. Math. Res. Not. IMRN 3 (2008).

[Ma97] B.Mazur, An introduction to the deformation theory of Galois representations, Modular forms and Fermat's last theorem, Springer Verlag (1997), 243-311.

[Na09] K.Nakamura, Classification of two dimensional split trianguline representations of p-adic fields, Compositio Math. 145 (2009), 865-914.

[Na11] K.Nakamura, Zariski density of crystalline representations for any p-adic field, preprint, arXiv:1104.1760.

[Pa10] V.Paskunas, The image of Colmez's Montreal functor, preprint, arXiv:1005.2008v1.

[Schl68] M.Schlessinger, Functors on Artin rings, Trans. A.M.S. 130 (1968), 208-222.

[Schn01] P.Schneider, Nonarchimedean Functional Analysis, Springer Monographs in Mathematics (2001).

[Se73] S.Sen, Lie algebras of Galois groups arising from Hodge-Tate modules, Ann. of Math. 97 (1) (1973), 160-170.